One-year premium risk and emergence pattern of ultimate loss based on conditional distribution *

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Abstract: We study the relation between one-year premium risk and ultimate premium risk. In practice, the one-year risk is usually related to the ultimate risk by using a so-called emergence pattern formula introduced by England et al. (2012) and Bird, Cairns (2011). We postulate to define the emergence pattern of the ultimate loss based on the conditional distribution of the best estimate of the ultimate loss given the ultimate loss, where the conditional distribution is derived from the multivariate distribution of the claims development process. We start with investigating Gaussian Incremental Loss Ratio, Hertig's Lognormal and Over-Dispersed Poisson claims development models. We derive the true emergence pattern formulas in these models and prove that they are different from the emergence pattern postulated by England et al. (2012), Bird, Cairns (2011). We assume that the risk is measured with Value-at-Risk. We identify that the true one-year risk can be significantly under and overestimated if the emergence pattern formula from England et al. (2012), Bird, Cairns (2011) is applied. We show that the ratio of the true one-year risk to the ultimate risk varies across the claims development models and depends on the confidence level. We prove that the one-year risk is lower than the ultimate risk only if a sufficiently high confidence level is used. Moreover, in a general claims development model we illustrate that the one-year risk can be higher than the ultimate risk at all high confidence levels and the distributions of the one-year risk and the ultimate risk can have different tail behaviour.

Keywords: One-year risk, ultimate risk, emergence pattern, Solvency II, claims development process.

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1 Introduction

Insurance companies are exposed to *premium and reserve risk* and they quantify so-called *one-year and ultimate risk*. By one-year premium risk we understand the risk that the premiums collected in a given year are not sufficient to cover the losses paid in the first year and the reserve set at the end of the first year for the outstanding losses (the one-year loss). By ultimate premium risk we understand the risk that the premiums earned in a given year are not sufficient to cover all losses paid in infinite time horizon (the ultimate loss). One-year and ultimate reserve risks are defined analogously, but the reserve risk is related to the adequacy of the current volumes of the claims reserves for the premiums already earned in the past. One-year premium and reserve risks and ultimate premium and reserve risks are investigated by actuaries for different purposes: the one-year risk is investigated for Solvency II risk capital, whereas the ultimate risk is investigated for pricing. The natural question is what is a relation between the one-year risk and the ultimate risk and how to model this relation. We try to answer this question in this paper. We focus on the premium risk, but our ideas can also be applied to the reserve risk.

Let $(X_1, X_2, ..., X_n)$ describe the claims development process for a new accident year. The random variable X_1 denotes the claims paid in the first development year and X_n denotes the aggregate value of all claims paid. The ultimate premium risk is modelled with X_n. The one-year premium risk is modelled with $BE_1 = \mathbb{E}[X_n|X_1]$. The expected value $BE_1 = \mathbb{E}[X_n|X_1]$ is called the best estimate of the ultimate loss after the first year. We can see that the relation between the one-year premium risk and the ultimate premium risk depends on the relation between X_1 and X_n. The relation between X_1 and X_n is usually specified in a reserve risk model which describes the claims development process $(X_1, ..., X_n)$. In a classical reserve risk model, see e.g. Radtke et al. (2016), Hertig (1985), Wüthrich, Merz (2008), we choose the distribution X_1 and the conditional distributions $(X_{i+1}|X_i)_{i=1}^{n-1}$. Having these distributions, the joint distribution $(X_1, ..., X_n)$ is defined, and consequently, the joint distribution of the one-year risk and the ultimate risk (BE_1, X_n) is defined. We can model the one-year risk and the ultimate risk (BE_1, X_n) by generating them in a forward way: we generate the first year loss X_1, calculate the best estimate $\mathbb{E}[X_n|X_1]$, using the development factors $(X_{i+1}|X_i)_{i=1}^{n-1}$ we generate the losses in the next development years and we get the ultimate loss X_n. The forward simulation of (BE_1, X_n) is clear and we know how to derive the ultimate loss X_n from the best estimate BE_1. The new problem which we investigate in this paper is how to model the one-year risk and the ultimate risk (BE_1, X_n) by generating them in a backward way starting with the ultimate loss X_n. Such a backward simulation scheme would be useful for investigating the one-year premium risk in Solvency II.

Insurance companies have put a lot of effort in estimating the distributions of the ultimate losses in their lines of business. From business point of view, the unconditional distribution of X_n is well-understood by decision makers, used in all planning reports, is the basis of pricing and long-term risk analysis. At the same time, the premium risk module in Solvency II focuses on the risk in the first year for the premiums earned, hence the distribution of X_1 must be used. Due to the business reasons mentioned above, insurance companies are less willing to use the distribution of X_1. In practice, insurance companies would usually prefer to perform the backward simulation of (X_1, X_n), i.e simulate the ultimate loss from a well-understood unconditional distribution of X_n and allocate the simulated ultimate loss X_n to the first development year with some intuitive allocation rule. From mathematical point of view, this allocation should be done based on the conditional distribution of $X_1|X_n$. More precisely, we should use the conditional distribution of $BE_1|X_n$ since BE_1 describes the one-year premium risk in Solvency II. In this paper, first of all, we discuss how we should generate the best estimate of the ultimate loss BE_1 from the ultimate loss X_n, i.e. how we should construct the allocation rule $BE_1|X_n$. Secondly, we investigate properties of the one-year risk vs. the ultimate risk in various claims development models. This paper is motivated by a real problem faced by insurance companies in Solvency II regime. To the best of our knowledge, the one-year risk has not been studied in the actuarial literature from theoretical point of view.

A practical approach to backward simulation of the cumulative payments $(X_1, ..., X_n)$ starting from the ultimate loss X_n is discussed in England et al. (2012) and Bird, Cairns (2011), who introduce concepts of *an emergence pattern* and *emergence factors*. The authors suggest that the ultimate loss X_n can be mapped into the best estimate of the ultimate loss BE_1 by using a simple linear function. To our knowledge, this allocation rule is used in practice by many insurance companies. In this paper we provide probabilistic foundations for the backward simulation of BE_1 starting from X_n. We postulate a new approach to define the emergence pattern of the ultimate loss where we use the multivariate distribution of the multi-period development of losses. More precisely, the emergence pattern formula is based on the conditional distribution of the best estimate of the ultimate loss given the ultimate loss $BE_1|X_n$ where the conditional distribution of $BE_1|X_n$ is derived from the assumed multivariate distribution of the claims development process $(X_1, ..., X_n)$. From the conditional distribution of $BE_1|X_n$, we next derive the unconditional distribution of the best estimate of the ultimate loss BE_1 used for quantifying the (unconditional) one-year risk. The distributions of $BE_1|X_n$ and BE_1 are parameterised in the spirit of the emergence pattern formula from England et al. (2012), Bird, Cairns (2011) so that they only depend on the unconditional distribution of the ultimate loss X_n and a single coefficient, still called an emergence factor, which summarises the information about the claims development process.

We consider three well-known reserve risk models (Gaussian Incremental Loss Ratio, Hertig's Lognormal, and Over-Dispersed Poisson models) and we establish the conditional distributions of $BE_1|X_n$ and the unconditional distributions of BE_1 in these models. We prove that the true emergence patterns in these three models are different from the emergence pattern postulated by England et al. (2012), Bird, Cairns (2011). Next, we assume that the risk is measured with Value-at-Risk. We identify that the true one-year risk can be significantly under and overestimated if the emergence pattern formula from England et al. (2012), Bird, Cairns (2011) is applied. In Hertig's Lognormal and Over-Dispersed Poisson models we show that the true one-year risk can be higher than the ultimate risk if a low confidence level is used and the true one-year risk is lower than the ultimate risk only if a sufficiently high confidence level is used. Moreover, in Over-Dispersed Poisson model the relation between the one-year risk and the ultimate risk can change at low confidence levels. We also prove that the ratio of the one-year risk to the ultimate risk for confidence levels $\gamma \to 1$ varies across models. Each of the three investigated reserve risk models assumes a particular distribution of the ultimate loss and a particular claims development process (respectively: normal, lognormal, and Poisson). In order to investigate possible relations between the one-year risk and the ultimate risk in various models (various distributions of the ultimate loss and various claims development processes), we keep the conditional distributions $BE_1|X_n$ from the well-known reserve risk models but use an arbitrary distribution X_n. Using this approach, we also gain flexibility in modelling the ultimate loss in the premium risk module beyond the distributions implied by the well-known reserve risk models. The key conclusions from this part of the paper are that the distribution of the one-year risk predicted by the emergence pattern formula by England et al. (2012), Bird, Cairns (2011) can have incorrect tail behaviour and the true one-year risk can be higher than the ultimate risk at all high confidence levels. The last result contradicts the common belief among actuaries that the one-year risk is always lower than the ultimate risk if we go far in the right tail, see e.g. Lloyd's (2014) or AISAM-ACME (2007).

In the sequel, we consider light-tailed, subexponential and Pareto-type distributions, see e.g. Chapters 2.2.3 and 2.5.2 in Rolski et al. (2001) for definitions. Since subexponential distributions include Pareto-type distributions, we focus on subexponential distributions with all moments finite. Unless otherwise stated, we assume that the distribution of the ultimate loss F is absolutely continuous with infinite right-end point and finite second moment.

By the true emergence pattern of the ultimate loss X_n we understand the conditional distribution of the best estimate of the ultimate loss given the ultimate loss $BE_1|X_n = x$ in the claims development model considered. We always assume that the risk is measured with Value-at-Risk. We quantify the (unconditional) ultimate risk and the (unconditional) one-year risk with $VaR_\gamma[X_n - \mathbb{E}[X_n]]$ and $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]$ at some confidence level γ. We always consider confidence levels $\gamma > 0.5$ such that $VaR_\gamma[X_n] > \mathbb{E}[X_n]$. Lower confidence levels are not relevant for practice. We remark that we may have $VaR_\gamma[BE_1] < \mathbb{E}[BE_1]$.

This paper is structured as follows. In Section 2 we describe the emergence pattern formula from England et al. (2012), Bird, Cairns (2011). In Section 3 we investigate the true emergence patterns and the true one-year risks in Gaussian Incremental Loss Ratio, Hertig's Lognormal and Over-Dispersed Poisson models. In Section 4 we study the one-year premium risk if we use an arbitrary distribution of the ultimate loss and allocate this loss to the first year by using the conditional distributions from Section 3. All proofs are presented in the last section. More numerical examples can be found in Szatkowski (2019).

2 The linear emergence pattern

England et al. (2012) and Bird, Cairns (2011) introduced the concept of an emergence pattern of the ultimate loss. They postulate the following linear relation between the best estimate of the ultimate loss and the ultimate loss:

$$BE_1^{ep} = \alpha X_n + (1-\alpha)\mathbb{E}[X_n], \tag{2.1}$$

where α is called an emergence factor, and $\alpha \in (0,1)$. England et al. (2012), Bird, Cairns (2011) propose to use α such that the standard deviations of BE_1^{ep} and BE_1 are equal. The emergence factor α is then given by

$$\alpha = \frac{SD[BE_1]}{SD[X_n]} = \frac{SD\big[BE_1 - \mathbb{E}[BE_1]\big]}{SD\big[X_n - \mathbb{E}[X_n]\big]}. \tag{2.2}$$

The emergence factor α measures the relation between the one-year risk and the ultimate risk, where the risk is measured with standard deviation. The parameter α is calibrated in a reserve risk model, see England et al. (2012), Bird, Cairns (2011).

We have a simple algorithm (2.1) for the simulation of BE_1 starting from X_n. Moreover, the parametrisation of the formula (2.1) is very appealing. We can use any distribution of the ultimate loss X_n parameterised in the premium risk model and we can switch from the ultimate premium risk to the one-year premium risk using a simple scaling factor α parameterised outside the premium risk model. The distribution of X_n does not have to be related to the distribution of the ultimate loss from the reserve risk model where the parameter α is calibrated.

Let us investigate theoretical foundations of the emergence pattern formula (2.1). The goal is to model a pair of dependent random variables (BE_1, X_n). The simplest approach would be to use a linear factor model for (BE_1, X_n). It is natural to assume that $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$, since otherwise the estimation of the ultimate loss is biased. We get the relation:

$$BE_1^f = \eta X_n + (1-\eta)\mathbb{E}[X_n] + \epsilon, \quad \eta = \rho(BE_1, X_n)\frac{SD(BE_1)}{SD(X_n)}, \tag{2.3}$$

where $\mathbb{E}[\epsilon] = 0, Var[\epsilon] = \big(1 - \rho^2(BE_1, X_n)\big)Var[BE_1]$ and $\epsilon \perp\!\!\!\perp X_n$. We can see that the linear factor model (2.3) leads to the emergence pattern (2.1) with α from (2.2) only if $\rho(BE_1, X_n) = 1$. We can immediately derive the following conclusions.

Corollary 2.1. *1. The emergence pattern formula* (2.1) *is built on the assumption that the best estimate of the ultimate loss BE_1 is perfectly linearly correlated with the ultimate loss X_n,*

2. The emergence pattern formula (2.1) *cannot describe the true emergence pattern of the ultimate loss, in the sense of the conditional distribution of $BE_1|X_n = x$, and the true joint distribution of the one-year risk and the ultimate risk* (BE_1, X_n) *in non-trivial models of claims development, including Gaussian Incremental Loss Ratio, Hertig's Lognormal, and Over-Dispersed Poisson models.*

The emergence pattern formula (2.1) implies that the conditional distribution $BE_1^{ep}|X_n = x$ is degenerate. However, the true conditional distribution $BE_1|X_n = x$ and the true conditional emergence pattern formula for the ultimate loss cannot be degenerate in a non-trivial claims development model, since we expect that there are many scenarios with many different possible values of best estimates of the ultimate loss which can lead to the same value of the ultimate loss.

In many applications, including Solvency II calculation of the capital requirement and quantification of the one-year premium risk, we only need an unconditional sample of the best estimate of the ultimate loss BE_1, instead of a joint sample (BE_1, X_n). Despite the failure of the formula (2.1) is modelling the true conditional distribution of $BE_1|X_n = x$, we can use the formula (2.1) to generate the unconditional sample of BE_1. We are now interested in the properties of the (unconditional) one-year risk vs. the ultimate risk.

Theorem 2.1. *Let us consider the emergence pattern formula* (2.1) *with an emergence factor* $\alpha \in (0, 1)$.

1. $\mathbb{E}[BE_1^{ep}] = \mathbb{E}[X_n]$ *and* $Var[BE_1^{ep}] = \alpha^2 Var[X_n] < Var[X_n]$,

2. If X_n has a light-tailed distribution (subexponential with all moments finite), then BE_1^{ep} has a light-tailed distribution (subexponential with all moments finite),

3. If X_n has a Pareto-type distribution with tail index θ, then BE_1^{ep} has a Pareto-type distribution with tail index θ,

4. $VaR_\gamma\big[BE_1^{ep} - E[BE_1^{ep}]\big] = \alpha VaR_\gamma\big[X_n - E[X_n]\big] < VaR_\gamma\big[X_n - E[X_n]\big]$.

If we use the emergence pattern formula (2.1), then the unconditional expected value of the best estimate of the ultimate loss is equal to the unconditional expected value of the ultimate loss. This property is desired since otherwise the estimation of the ultimate loss is biased. If the emergence factor α is set in accordance with condition (2.2), then $Var[BE_1^{ep}] = Var[BE_1]$, which is again a desired property. Finally, the one-year risk is a fraction of order α of the ultimate risk, if the risk is measured with Value-at-Risk. This property is less clear. However, there is a common belief among actuaries that the one-year risk is lower than the ultimate risk, see e.g. Lloyd's (2014) or AISAM-ACME (2007) for arguments supporting this statement. The results from Theorem 2.1 can be reformulated as the corollary.

Corollary 2.2. *Let us consider the emergence pattern formula* (2.1) *with* $\alpha \in (0, 1)$.

1. *The one-year risk is lower than the ultimate risk at all confidence levels,*
2. *The one-year risk decreases linearly in* α *at all confidence levels when the emergence factor* α *decreases,*
3. *The distributions of the one-year risk and the ultimate risk have the same tail behaviour.*

Although the emergence pattern formula (2.1) cannot properly describe the emergence pattern of the ultimate loss in the sense of the conditional distribution of $BE_1|X_n = x$, Theorem 2.1 and Corollary 2.2 show that the emergence pattern formula (2.1) has some desirable properties if applied to simulate the unconditional samples of the best estimates of the ultimate loss BE_1 from the ultimate losses X_n. However, at the same time we can expect that, in general, the true relation between $VaR_\gamma[BE_1]$ and $VaR_\gamma[X_n]$ is different from the relation α between standard deviations. Consequently, if we assume that $SD[BE_1] = \alpha SD[X_n]$, then the true relation between $VaR_\gamma[BE_1]$ and $VaR_\gamma[X_n]$ is unlikely to be also linear in α. Moreover, it is far from clear whether the one-year risk is indeed always lower than the ultimate risk at all confidence levels, if measured with Value-at-Risk. It is also far from clear how the tail of the distribution of the one-year risk is related to the tail of the distribution of the ultimate risk.

We shall be interested in deriving the true emergence pattern formulas and investigating the true relation between the one-year risk and the ultimate risk.

3 The true emergence patterns in reserve risk models

In the next subsections, we derive the true conditional distributions of $BE_1|X_n$ and the unconditional distributions of BE_1 in three well-known reserve risk models. We parameterise the distributions in the spirit of the emergence pattern formula (2.1).

3.1 Gaussian Incremental Loss Ratio model

We consider Incremental Loss Ratio model (ILR) with Gaussian incremental losses, see e.g. Radtke et al. (2016) and Wüthrich, Merz (2008). We investigate the cumulative payments described by the model:

$$X_j = \sum_{i=1}^{j} \epsilon_i, \qquad \text{where:} \quad \epsilon_i \sim N(Em_i; E\sigma_i^2) \quad \text{for } i \in \{1, ..., n\},$$
$$\text{and:} \quad \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } i \neq j \in \{1, ..., n\}. \tag{3.1}$$

E denotes the exposure in the accident year, and ϵ_i represents the incremental loss in development year i. We assume that $m_i \in \mathbb{R}, \sigma_i > 0, E > 0$. We denote

$$m = \sum_{i=1}^{n} m_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In this reserve risk model, the best estimate of the ultimate loss is given by

$$BE_1 = \mathbb{E}[X_n|X_1] = X_1 + E(m - m_1). \tag{3.2}$$

Proposition 3.1. *Let us consider the model* (3.1) *of claims development and the multivariate distribution of the claims development process* $(X_1, ..., X_n)$. *We have the following loss distributions:*

$$X_n \sim N(Em; E\sigma^2), \tag{3.3}$$

$$BE_1|X_n = x \sim N\left(\frac{\sigma_1^2}{\sigma^2}x + \frac{\sigma^2 - \sigma_1^2}{\sigma^2}Em; E\frac{\sigma_1^2(\sigma^2 - \sigma_1^2)}{\sigma^2}\right), \tag{3.4}$$

$$BE_1 \sim N(Em; E\sigma_1^2). \tag{3.5}$$

The proposition above is derived by using all available information from the reserve risk model (3.1) where the claims development dynamics and the distributions of the incremental losses in the consecutive periods are fully specified. When the actuary generates the conditional and the unconditional best estimate of the ultimate loss after the first year for the one-year premium risk in Solvency II, they would prefer to use the information about the ultimate loss from the premium risk model and limited information from the reserve risk model. Hence, we try to reparameterise the distributions from Proposition 3.1 so that they only depend on the unconditional distribution of the ultimate loss and a proportion of the one-year risk to the ultimate risk. We measure the proportion of the one-year risk to the ultimate risk with standard deviation as in (2.2). We follow the main idea behind the emergence pattern formula (2.1). However, we are able to improve the emergence pattern formula from England et al. (2012), Bird, Cairns (2011) so that it yields the correct conditional distribution of $BE_1|X_n$ and the correct unconditional distribution of X_n in our reserve risk model.

Theorem 3.1. *Let us consider the model* (3.1) *of claims development and the multivariate distribution of the claims development process* $(X_1, ..., X_n)$*. Let*

$$\mu_{X_n} = \mathbb{E}[X_n], \quad \sigma^2_{X_n} = Var[X_n], \quad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We have the following loss distributions:

$$X_n \sim N\Big(\mu_{X_n}; \sigma^2_{X_n}\Big), \tag{3.6}$$

$$BE_1|X_n = x \sim N\Big(\alpha^2 x + (1-\alpha^2)\mu_{X_n}; \alpha^2(1-\alpha^2)\sigma^2_{X_n}\Big), \tag{3.7}$$

$$BE_1 \sim N\Big(\mu_{X_n}; \alpha^2\sigma^2_{X_n}\Big), \tag{3.8}$$

with the parameter $\alpha \in (0, 1)$.

Theorem 3.1 establishes a backward simulation scheme for BE_1 starting from X_n where we can switch (in accordance with the underlying probabilistic model of claims development) from the ultimate risk to the one-year risk. As in (2.1), we only use the distribution of the ultimate loss X_n from the ultimate premium risk and a single emergence factor α which summarises the information from the reserve risk module about the claims development process required for the one-year premium risk. We remark that the distribution of the development factor $X_n|BE_1 = b$ can be parameterised in the similar fashion, hence we also have a forward simulation scheme in our framework where we can we can switch from the one-year risk to the ultimate risk, see Szatkowski (2019).

Both formulas (3.7) and (2.1) allow us to map the ultimate loss to the best estimate of the ultimate loss after the first year. The formulas are different since they are derived from different assumptions. We point out that only the formula (3.7) gives the true emergence pattern of the ultimate loss in our the claims development model (3.1). The conditional distribution $BE_1^{ep}|X_n = x$ implied by the emergence pattern (2.1) is degenerate and fails to describe the true probabilistic relation between (BE_1, X_n). The conditional distribution $BE_1|X_n = x$ derived from the claims development process (3.1) is not degenerate and captures the true probabilistic relation between (BE_1, X_n). This property has already been observed in Corollary 2.1.

If we assume, as in Theorem 3.1, that the ultimate loss $X_n \sim N(\mu_{X_n}, \sigma^2_{X_n})$, then the emergence pattern formula (2.1) yields the unconditional distribution of the best estimate of the ultimate loss:

$$BE_1^{ep} \sim N\Big(\mu_{X_n}, \alpha^2\sigma^2_{X_n}\Big). \tag{3.9}$$

Interestingly, the unconditional distributions of the best estimate of the ultimate loss (3.8) and (3.9) generated with the formulas (3.7), (2.1) are the same.

We can state properties of the unconditional distribution of the best estimate of the ultimate loss in our model, which are special cases of the properties from Theorem 2.1.

Theorem 3.2. *Let us consider the model and the assumptions from Theorem 3.1.*

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,
2. $VaR_\gamma\big[BE_1 - E[BE_1]\big] = \alpha VaR_\gamma\big[X_n - E[X_n]\big] < VaR_\gamma\big[X_n - E[X_n]\big]$, *where*

$$VaR_\gamma\big[BE_1 - E[BE_1]\big] = \alpha\Phi^{-1}(\gamma)SD[X_n].$$

We conclude with a corollary with properties of the one-year risk.

Corollary 3.1. *Let us consider the model and the assumptions from Theorem 3.1.*

1. *The emergence pattern formula* (2.1) *yields the proper distribution of the one-year risk* (3.8)*,*
2. *Corollary 2.2 holds.*

We can conclude that there exists a non-trivial claims development model where the unconditional distribution of the one-year risk is described with the emergence pattern formula from England et al. (2012), Bird, Cairns (2011). However, this is a special case.

3.2 Hertig's Lognormal model

We consider a multiplicative loss model where the development factors are modelled with lognormal distributions, see e.g. Hertig (1985) or Wüthrich, Merz (2008). We investigate the cumulative payments described by the model:

$$X_1 = \epsilon_1, \quad X_j = X_{j-1}\cdot\epsilon_j, \qquad \text{where: } \epsilon_i \sim LogN(m_i, \sigma_i^2) \quad \text{for } i \in \{1,...,n\},$$
$$\text{and: } \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } i \neq j \in \{1,...,n\}. \tag{3.10}$$

We assume that $m_i \in \mathbb{R}, \sigma_i > 0$. We denote

$$m = \sum_{i=1}^{n} m_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In this reserve risk model, the best estimate of the ultimate loss is given by

$$BE_1 = \mathbb{E}[X_n|X_1] = X_1 e^{m - m_1 + \frac{1}{2}(\sigma^2 - \sigma_1^2)}. \tag{3.11}$$

Using the same reasoning as in the previous section, we can derive the true emergence pattern for the ultimate loss which comes from the claims development process (3.10).

Theorem 3.3. *Let us consider the model* (3.10) *of claims development and the multivariate distribution of the claims development process* $(X_1, ..., X_n)$. *Let*

$$\mu_{X_n} = \mathbb{E}[X_n], \quad \psi_{X_n} = \frac{SD[X_n]}{E[X_n]}, \quad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We have the following loss distributions:

$$X_n \sim LogN\Big(\tilde{m}; \tilde{\sigma}^2\Big), \tag{3.12}$$

$$BE_1 | X_n = x \sim LogN\left(\tilde{\alpha}^2 \log(x) + \Big(1 - \tilde{\alpha}^2\Big)\Big(\tilde{m} + \frac{\tilde{\sigma}^2}{2}\Big); \tilde{\alpha}^2(1 - \tilde{\alpha}^2)\tilde{\sigma}^2\right), \tag{3.13}$$

$$BE_1 \sim LogN\Big(\tilde{m} + (1 - \tilde{\alpha}^2)\frac{\tilde{\sigma}^2}{2}; \tilde{\alpha}^2\tilde{\sigma}^2\Big), \tag{3.14}$$

where

$$\tilde{m} = log(\mu_{X_n}) - \frac{1}{2}\log\big(1 + \psi^2_{X_n}\big), \quad \tilde{\sigma}^2 = \log\big(1 + \psi^2_{X_n}\big),$$
$$\tilde{\alpha}^2 = \frac{\log\big(1 + \alpha^2\psi^2_{X_n}\big)}{\log\big(1 + \psi^2_{X_n}\big)},$$

with the parameter $\alpha \in (0, 1)$.

Theorem 3.4. *Let us consider the model and the assumptions from Theorem 3.3.*

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,

2. $VaR_\gamma\big[BE_1 - E[BE_1]\big] < VaR_\gamma[X_n - E[X_n]]$ *for* $\gamma > \gamma^*$, $VaR_\gamma\big[BE_1 - E[BE_1]\big] > VaR_\gamma[X_n - E[X_n]]$ *for* $\gamma < \gamma^*$, *where*

$$VaR_\gamma[BE_1] = \mu_{X_n}\big(1 + \alpha^2\psi^2_{X_n}\big)^{-1/2} e^{\sqrt{\log\left(1+\alpha^2\psi^2_{X_n}\right)}\Phi^{-1}(\gamma)},$$
$$\gamma^* = \Phi\Big(\frac{1}{2}\sqrt{\log(1 + \alpha^2\psi^2_{X_n})} + \frac{1}{2}\sqrt{\log(1 + \psi^2_{X_n})}\Big),$$

3. *For any* $k \in \mathbb{N}$, $VaR_\gamma\big[BE_1 - E[BE_1]\big] < \alpha^k VaR_\gamma[X_n - E[X_n]]$ *for all* $\alpha \in [\alpha_0, \alpha_1] \subset (0, 1)$ *and* $\gamma > \gamma^*$, *where*

$$\gamma^* = \Phi\left(\Big(\sqrt{\log(1 + \alpha_1^2\psi^2_{X_n})} + \sqrt{\log(1 + \psi^2_{X_n})}\Big) \cdot \Big(\frac{1}{2} + k\frac{-\log\alpha_0}{\log\left(\frac{1+\psi^2_{X_n}}{1+\alpha_1^2\psi^2_{X_n}}\right)}\Big)\right).$$

   In addition, we have the limit

$$\lim_{\gamma\to 1}\frac{VaR_\gamma\big[BE_1 - \mathbb{E}[BE_1]\big]}{VaR_\gamma\big[X_n - \mathbb{E}[X_n]\big]} = 0.$$

If we make the assumptions from Theorem 3.3 and apply the emergence pattern formula (2.1) to the claims development process implied by the reserve risk model (3.10), then the unconditional distribution of the best estimate of the ultimate loss is given by

$$BE_1^{ep} \sim \alpha \cdot LogN(\tilde{m}, \tilde{\sigma}^2) + (1-\alpha)\mu_{X_n}. \tag{3.15}$$

Let us now compare Theorem 3.4 with 2.1 and the emergence pattern formula (3.13) with (2.1). Recalling the discussion from the previous sections, it is clear that the conditional distributions of the best estimate of the ultimate loss given the ultimate loss (3.13) and (2.1) are different. We can now observe that the unconditional distributions of the best estimate of the ultimate loss (3.14) and (3.15) are also different if derived from the emergence pattern formula and the conditional distribution approach. Of course, only the formula (3.13) and the distribution (3.14) characterise the true emergence pattern of the ultimate loss in the claims development model (3.10).

We now focus on the one-year risk. We can notice that the supports of BE_1 and BE_1^{ep} generated with (3.14) and (3.15) are different. From point 2 from Theorem 3.4 we deduce that the one-year risk can be higher than the ultimate risk in the claims development model (3.10) if the confidence level γ is too low. We point out that a confidence level γ at which we may observe that $VaR_\gamma[BE_1] > VaR_\gamma[X_n]$ is above the lowest confidence level we allow for, i.e. is above $\tilde{\gamma} = \Phi\big(\frac{1}{2}\sqrt{\log(1+\psi_{X_n}^2)}\big)$ at which $VaR_{\tilde{\gamma}}[X_n] = E[X_n]$. For a very skewed distribution of X_n with high ψ_{X_n}, a *low* confidence level γ at which we may observe that $VaR_\gamma[BE_1] > VaR_\gamma[X_n]$ can be very high, if α is also high. Hence, at confidence levels relevant in practice the one-year risk can be higher than the ultimate risk. However, at all sufficiently high confidence levels the one-year risk is lower then the ultimate risk in the claims development model (3.10). This property stands in contrast with the property of the emergence pattern formula (2.1) where the one-year risk is lower than the ultimate risk at all confidence levels. From point 3 from Theorem 3.4 we also conclude that, at sufficiently high confidence levels, the true ratio between the Value-at-Risk for the one-year risk BE_1 and the Value-at-Risk for the ultimate risk X_n decreases faster than any polynomial in α when the emergence factor α decreases (whereas the ratio between the Value-at-Risk for BE_1^{ep} and the Value-at-Risk for X_n is linear in α). Summing up, we can conclude that the emergence pattern formula (2.1) underestimates the true one-year risk at low confidence levels and overestimates the true one-year risk at high confidence levels. The above properties are illustrated in the numerical example below.

Example 3.1. We choose $\mu_{X_n} = 1$ since the expected value μ_{X_n} is scaling parameter for X_n. We calculate the ratios of $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ as a function of α, for different ψ_{X_n} and γ. Let us recall that $\frac{VaR_\gamma[BE_1^{ep} - \mathbb{E}[BE_1^{ep}]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]} = \alpha$ for any ψ_{X_n} and γ. The results are presented in Figure 1. Our numerical results confirm our expectations. The emergence pattern formula (2.1) can significantly under and overestimate the true one-year risk in the claims development model (3.10). E.g. for $\psi_{X_n} = 1$ and $\alpha = 50\%$, the emergence pattern formula (2.1) overestimates the true one year risk at $\gamma = 99.5\%$ by about 25%. For $\psi_{X_n} = 3$ and $\alpha = 50\%$, the emergence pattern formula (2.1) underestimates the true one year risk at $\gamma = 99.5\%$ by about 9%. We can see that the discrepancies between the linear emergence pattern formula and the true one-year risk can be much larger in both directions for different parameters. The true one-year risk can be higher than the ultimate risk since we observe $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]} > 1$ at lower confidence levels γ. Even though point 3 from Theorem 3.4 shows that $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] = o\big(VaR_\gamma[X_n - \mathbb{E}[X_n]]\big)$ for $\gamma \to 1$, we end up with $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] \approx \alpha^2 VaR_\gamma[X_n - \mathbb{E}[X_n]]$ for γ in the range $99.99\% - 99.999\%$ in our numerical experiments. □

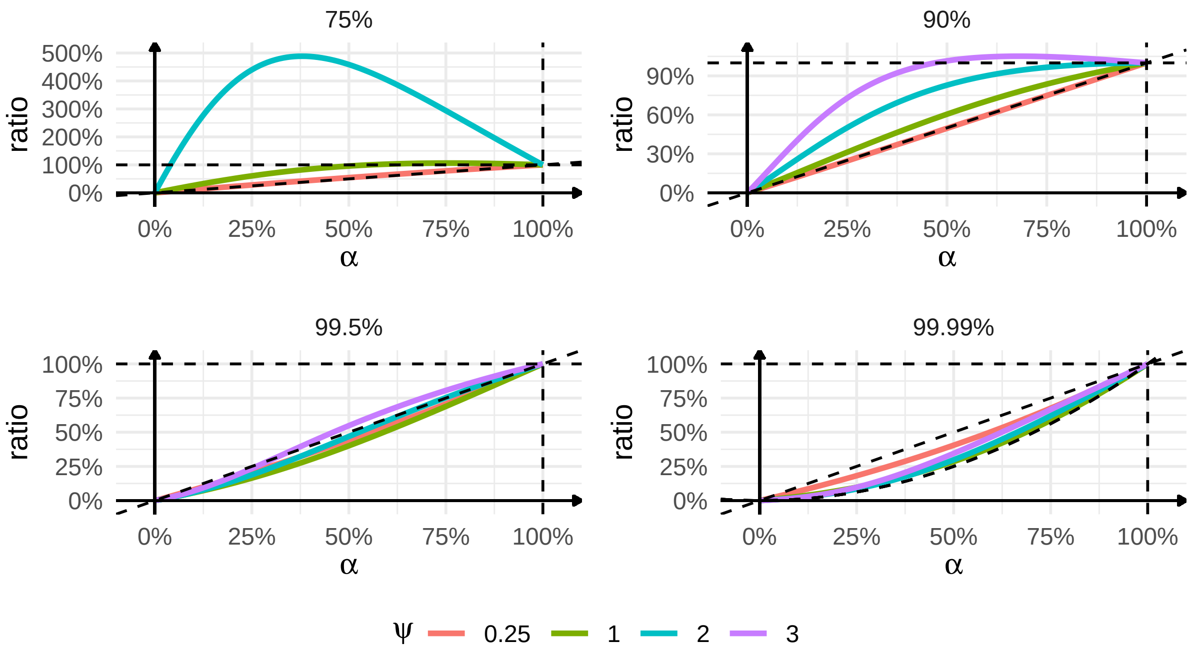

Figure 1: The ratios $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the Hertig's Lognormal model.

We can collect all our observations.

Corollary 3.2. *Let us consider the model and the assumptions from Theorem 3.3.*

1. *The emergence pattern formula* (2.1) *overestimates the true one-year risk at high confidence levels and underestimates the true one-year risk at low confidence levels,*

2. *The true one-year risk is lower than the ultimate risk at all high confidence levels and the true one-year risk is higher than the ultimate risk at all low confidence levels,*

3. The true one-year risk vanishes compared to the ultimate risk for the confidence level in the limit $\gamma \to 1$. In practical examples, the ratio between the true one-year risk and the ultimate risk at very high confidence levels can be of order α^2.

We can see that the true emergence pattern of the ultimate loss and the relation between the one-year risk and the ultimate risk in the claims development model (3.10) are much more sophisticated than the relations postulated by the simple linear emergence pattern formula from England et al. (2012), Bird, Cairns (2011).

3.3 Over-Dispersed Poisson model

Finally, we consider Incremental Loss Ratio model with Over-Dispersed Poisson incremental losses, see e.g. Wüthrich, Merz (2008) or Taylor (2016). We investigate the cumulative payments described by the model:

$$X_j = \sum_{i=1}^{j} \epsilon_i, \qquad \text{where:} \quad \epsilon_i \sim ODP\big(\mu\gamma_i, \phi\big) \quad \text{for } i \in \{1, ..., n\},$$
$$\text{and:} \quad \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } i \neq j \in \{1, ..., n\}. \tag{3.16}$$

We assume that $\mu > 0, \gamma_i > 0, \phi > 0$. The assumption that $\epsilon \sim ODPoiss\big(\lambda, \phi\big)$ means that $\epsilon/\phi \sim Poiss\big(\lambda/\phi\big)$. We denote

$$\gamma = \sum_{i=1}^{n} \gamma_i.$$

In this reserve risk model, the best estimate of the ultimate loss is given by

$$BE_1 = \mathbb{E}[X_n|X_1] = X_1 + \mu(\gamma - \gamma_1). \tag{3.17}$$

If $\phi = 1$, then the model (3.16) describes the development process of the number of claims incurred. The true emergence pattern of the ultimate loss which comes from the claims development process (3.16) is given in the next theorem.

Theorem 3.5. *Let us consider the model* (3.16) *of claims development and the multivariate distribution of the claims development process* $(X_1, ..., X_n)$. *Let*

$$\mu_{X_n} = \mathbb{E}[X_n], \quad \phi_{X_n} = \frac{Var[X_n]}{E[X_n]}, \quad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We have the following loss distributions:

$$X_n \sim \phi_{X_n} \cdot Poiss\big(\mu_{X_n}/\phi_{X_n}\big), \tag{3.18}$$
$$BE_1|X_n = x \sim \phi_{X_n} \cdot Bin\Big(\frac{x}{\phi_{X_n}}; \alpha^2\Big) + (1-\alpha^2)\mu_{X_n}, \tag{3.19}$$
$$BE_1 \sim \phi_{X_n} \cdot Poiss\big(\alpha^2\mu_{X_n}/\phi_{X_n}\big) + (1-\alpha^2)\mu_{X_n}, \tag{3.20}$$

with the parameter $\alpha \in (0, 1)$.

Theorem 3.6. *Let us consider the model and the assumptions from Theorem 3.5.*

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,

2. $VaR_\gamma\big[BE_1 - E[BE_1]\big] < VaR_\gamma[X_n - E[X_n]]$ *for* $\gamma > \gamma^*$ *where* $\gamma^* < 1$ *is the last point where the distributions of* BE_1 *and* X_n *cross, or* $VaR_\gamma\big[BE_1 - E[BE_1]\big] < VaR_\gamma[X_n - E[X_n]]$ *for all* γ *if the distributions of* BE_1 *and* X_n *don't cross. The relation between* $VaR_\gamma\big[BE_1 - E[BE_1]\big]$ *and* $VaR_\gamma[X_n - E[X_n]]$ *can change for* $\gamma < \gamma^*$ *since the distributions of* BE_1 *and* X_n *can cross more than once (if they cross),*

3. *We have the limit*

$$\lim_{\gamma \to 1} \frac{VaR_\gamma\big[BE_1 - \mathbb{E}[BE_1]\big]}{VaR_\gamma\big[X_n - \mathbb{E}[X_n]\big]} = 1.$$

If we make the assumptions from Theorem 3.5 and we apply the emergence pattern formula (2.1) to the claims development process implied by the reserve risk model (3.16), then the unconditional distribution of the best estimate of the ultimate loss is given by

$$BE_1^{ep} \quad \sim \quad \alpha\phi_{X_n} \cdot Poiss\big(\frac{\mu_{X_n}}{\phi_{X_n}}\big) + (1-\alpha)\mu_{X_n}. \tag{3.21}$$

The conditional and the unconditional distributions of $BE_1|X_n$ and BE_1 given by (3.19)-(3.20) and (2.1) are different if derived from the emergence pattern formula and the conditional distribution approach. Only the formula (3.19) and the distribution (3.20) characterise the true emergence pattern of the ultimate loss in the claims development model (3.16). The supports of BE_1 and BE_1^{ep} generated with (3.20) and (3.21) are different. By point 2 from Theorem 3.6, there exists a confidence level $\gamma^* < 1$ such that the one-year risk is lower than the ultimate risk at all high confidence levels $\gamma > \gamma^*$. At low confidence levels $\gamma < \gamma^*$, the one-year risk can be lower or higher than the ultimate risk. This property makes the interpretation of the one-year risk in relation to the ultimate risk difficult in Poisson models since the relation between the one-year risk and the ultimate risk can change multiple times as we change the confidence level. This property is illustrated in Example 3.2. Let us recall that in the Hertig's Lognormal model we have identified that the one-year risk is higher than the ultimate risk at all low confidence levels below some γ^* and the one-year risk is lower than the ultimate risk at all high confidence levels above γ^*. From point 3 from Theorem 3.6 we can deduce that the ratio between the Value-at-Risk measures for the one-year risk BE_1 and the ultimate risk X_n at a sufficiently high confidence level should decrease slower than linearly in α when the emergence factor α decreases. Consequently, the emergence pattern formula (2.1) underestimates the true one-year risk at all sufficiently high confidence levels. The emergence pattern formula (2.1) also underestimates the true one-year risk at some low confidence levels e.g. when the true one-year risk is above the ultimate risk.

Example 3.2. We choose $\psi_{X_n} = 1$ and we consider Poisson distributions of X_n with different λ. The Over-Dispersed Poisson model approaches the Gaussian Incremental Loss Ratio model for $\lambda \to \infty$, hence we are mainly interested in low λ. In order to better understand the crossing property of the distributions of BE_1 and X_n, we first investigate the distributions of BE_1 and X_n for $\lambda \in \{1.5, 25\}$ and $\alpha \in \{85\%, 15\%\}$, see Figure 2. For $\lambda = 25, \alpha = 15\%$, the distribution of BE_1 dominates the distribution of X_n over the whole range above the median and $VaR_\gamma[BE_1] < VaR_\gamma[X_n]$ for all $\gamma > 0.5$. However, for $\lambda = 1.5, \alpha = 85\%$, the distributions of BE_1 and X_n cross 9 times in the range above the median and the distribution of BE_1 dominates the distribution of X_n only for $x > 5.41625$. This implies that $VaR_\gamma[BE_1] < VaR_\gamma[X_n]$ only for $\gamma > 99.555\%$. For $\gamma < 99.555\%$, we may have $VaR_\gamma[BE_1] < VaR_\gamma[X_n]$ and $VaR_\gamma[BE_1] > VaR_\gamma[X_n]$ depending on γ. From (5.3) we can deduce that the threshold γ^* is high for high α and low λ, whereas the threshold γ^* is low for low α and high λ.

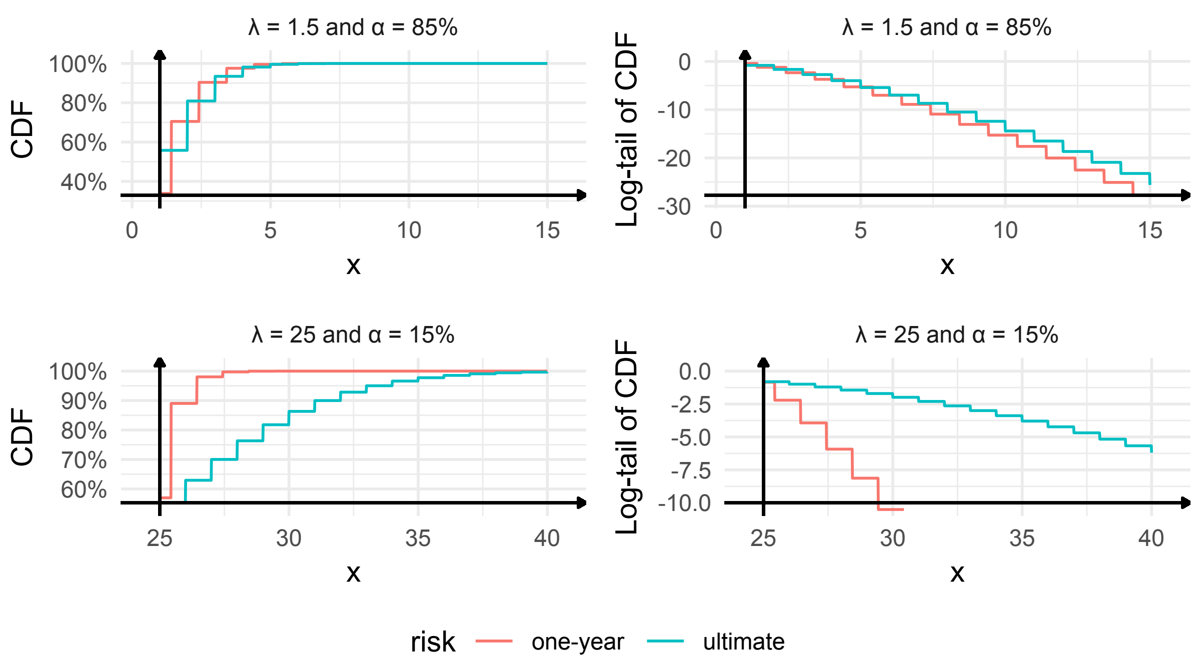

Figure 2: The distributions of BE_1 and X_n in the Poisson model.

Our key results are presented in Figure 3. We remark that some ratios are negative since $VaR_\gamma[BE_1] < \mathbb{E}[BE_1]$ (but $VaR_\gamma[X_n] > \mathbb{E}[X_n]$ by our assumption). We confirm that the emergence pattern formula (2.1) can significantly under and overestimate the true one-year risk in the claims development model (3.16). E.g. for $\lambda = 1.5$ and $\alpha = 85\%$, the emergence pattern formula (2.1) underestimates the true one-year risk at $\gamma = 99.5\%$ by 24% and the true one-year risk is above the ultimate risk by 12%. For $\lambda = 25$ and $\alpha = 15\%$, the emergence pattern formula (2.1) underestimates the true one-year risk at $\gamma = 99.5\%$ by 14%. For $\lambda = 1.5$ and $\alpha = 85\%$, the one-year risk is higher than the ultimate risk at $\gamma = 75\%$ and 99.5%, but the one-year risk is lower than the ultimate risk for $\gamma = 90\%$ and 99.99% - and we can observe the crossing property for $\gamma < 99.555\%$. Moreover, the emergence pattern formula (2.1) overestimates the true one-year risk at $\gamma = 90\%$ by 39%. Finally, even though point 3 from Theorem 3.6 shows that $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] \sim VaR_\gamma[X_n - \mathbb{E}[X_n]]$ for $\gamma \to 1$, the convergence for γ in the range $99.99\% - 99.999\%$ is slow in our numerical experiments, yet it can be observed. □

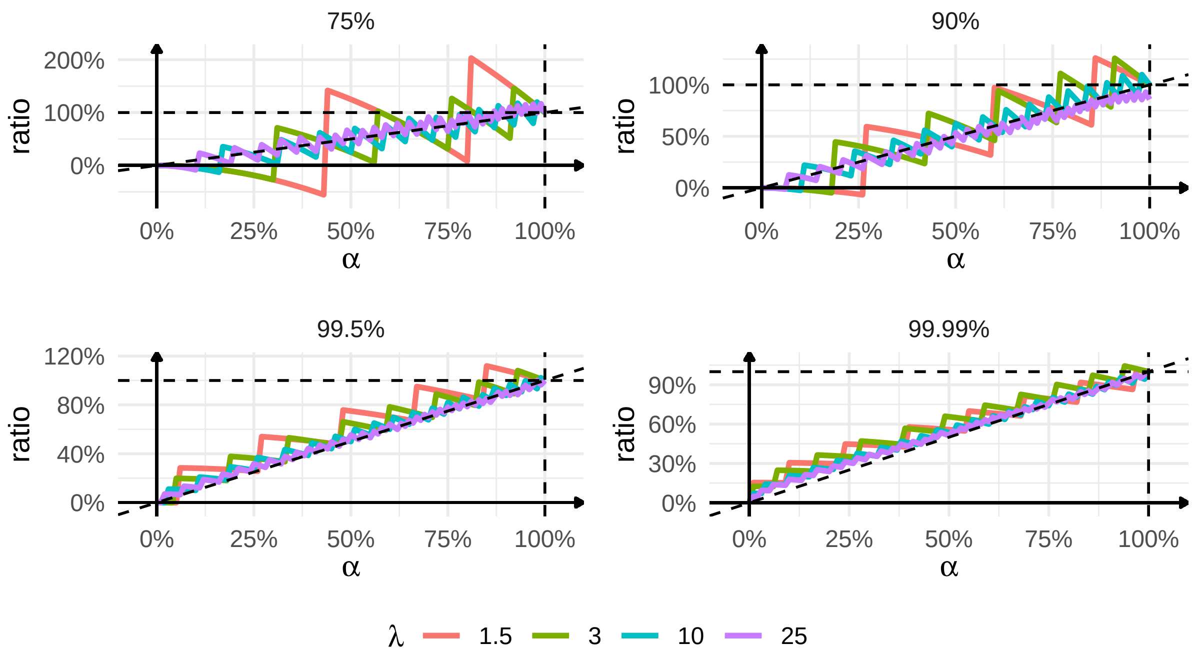

Figure 3: The ratios $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the Poisson model.

Corollary 3.3. *Let us consider the model and the assumptions from Theorem 3.5.*

1. *The emergence pattern formula* (2.1) *underestimates the true one-year risk at high confidence levels,*

2. *The true one-year risk is lower than the ultimate risk at all high confidence levels,*

3. *The emergence pattern formula* (2.1) *can overestimate and underestimate the true one-year risk and the true one-year risk can be higher and lower than the ultimate risk at low confidence levels,*

4. *The true one-year risk approaches the ultimate risk for the confidence level in the limit* $\gamma \to 1$*. In practical examples, the convergence is slow.*

4 One-year premium risk with an arbitrary ultimate loss distribution

The reserve risk models which we investigated in the previous section assume a particular form of the claims development process and lead to a particular distribution of the ultimate loss. The parametric distribution of the ultimate loss is fixed and is inherently related with the reserve risk model considered. What is important for practical applications in Solvency II in premium risk module is the possibility of using any distribution of the ultimate loss. Let us recall that we can use any distribution of the ultimate loss and apply the linear emergence pattern formula (2.1).

It may be difficult to specify a priori the claims development process and the joint multivariate model for cumulative payments $(X_1, ..., X_n)$ which lead to a pre-specified distribution of the ultimate loss X_n. Models of claims development are developed in the literature on claims reserving and the range of possible distributions of the ultimate loss in these models is limited. At the same time, the range of distributions used in premium risk is large. What we suggest is to use an arbitrary unconditional distribution of the ultimate loss X_n and a conditional distribution of $BE_1|X_n$, extracted from the well-known reserve risk model, as a mechanism for allocating the ultimate loss X_n to the best estimate BE_1. Of course, with this approach we change the joint distribution of (BE_1, X_n) in the original reserve risk model from which the distribution of $BE_1|X_n$ is extracted.

Let us consider a pair of dependent random variables $(BE_1, X_n) \sim f$ with the marginal and conditional distributions:

$$X_n \sim f_{X_n}, \quad X_n|BE_1 = x \sim f_{X_n|BE_1=x}, \quad BE_1|X_n = x \sim f_{BE_1|X_n=x}.$$

We define a new bivariate distribution $(BE_1, X_n) \sim f^{new}$ by choosing $BE_1|X_n = x \sim f_{BE_1|X_n=x}$ and $X_n \sim f^{new}_{X_n}$. We have the marginal and conditional distributions:

$$\begin{aligned} f^{new}_{BE_1}(z) &= \int f_{BE_1|X_n=x}(z) f^{new}_{X_n}(x)dx, \\ f^{new}_{X_n|BE_1=x}(z) &= C \cdot f_{X_n|BE_1=x}(z) \frac{f^{new}_{X_n}(z)}{f_{X_n}(z)}, \end{aligned}$$

where C denotes a normalising constant which depends on x but is independent of z. See Example 4.1 below and Szatkowski (2019).

We can now construct new models of claims development by using the distributions of the ultimate loss X_n and the emergence pattern $BE_1|X_n$. In contrast to (2.1), the emergence pattern $BE_1|X_n$ is no longer a linear function of the ultimate loss X_n but a random function of the ultimate loss X_n which distribution depends on X_n. Our approach has two advantages. Firstly, we have a flexible and interpretable probabilistic model where we can freely choose the distribution of the ultimate loss and switch from the ultimate risk to the one-year risk, and vice verse. This is a desired feature in Solvency II premium risk modelling. Secondly, we can investigate properties of the one-year risk vs. the ultimate risk in various claims development models, beyond the models we know from the claims reserving literature. This is our main goal in the next sections.

4.1 The conditional distribution from Gaussian ILR model

From (3.7) we can conclude that the unconditional distribution of the best estimate of the ultimate loss in the model (3.1) is a mixture of normal distributions with an uncertain expected value. Consequently, we have the representation:

$$BE_1 = \alpha^2 X_n + (1-\alpha^2)\mu_{X_n} + \sqrt{\alpha^2(1-\alpha^2)}\sigma_{X_n}\xi, \tag{4.1}$$

where $\xi \sim N(0,1)$, and ξ is independent of X_n. The formula (4.1) is our new emergence pattern formula which allocates the ultimate loss with an arbitrary distribution to the best estimate of the ultimate loss. We can call (4.1) *the additive normal emergence pattern formula*. The additive normal emergence pattern can be seen as a version of the classical emergence pattern formula (2.1) where we simply add a Gaussian noise in order to have a non-degenerate distribution of $BE_1|X_n = x$.

Theorem 4.1. *Let us assume that the ultimate loss is modelled with $X_n \sim F_{X_n}$. Let $\mu_{X_n} = E[X_n]$, $\sigma^2_{X_n} = Var[X_n]$ and choose an emergence factor $\alpha \in (0,1)$. The emergence pattern of the ultimate loss X_n is described with the formula (4.1). We have the following properties:*

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,
2. *If X_n has a light-tailed distribution (subexponential with all moments finite), then BE_1 has a light-tailed distribution (subexponential with all moments finite). Moreover, if X_n has a light-tailed distribution such that $\lim_{x\to\infty}(1-F_{X_n}(x))e^{vx^2} = 0$ for all $v > 0$, then $VaR_\gamma\big[BE_1 - \mathbb{E}[BE_1]\big] > VaR_\gamma\big[X_n - \mathbb{E}[X_n]\big]$ for $\gamma > \gamma^*$, with some $\gamma^* < 1$,*
3. *If X_n has a Pareto-type distribution with tail index θ, then BE_1 has a Pareto-type distribution with tail index θ, and we have the limit*

$$\lim_{\gamma\to 1}\frac{VaR_\gamma\big[BE_1 - E[BE_1]\big]}{VaR_\gamma\big[X_n - E[X_n]\big]} = \alpha^2.$$

We have already observed in the previous sections that $\lim_{\gamma\to 1}\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ varies across claims development models and may not be equal to α, see Theorems 3.2, 3.4, 3.6. Point 3 from Theorem 4.1 gives a new possible value of $\lim_{\gamma\to 1}\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$, which is different from the limits proved in the previous sections. More importantly, from point 2 from Theorem 4.1 we deduce a new and interesting property that the one-year risk can be higher than the ultimate risk at all high confidence levels. This property of the one-year risk vs. the ultimate risk is not observed in the linear emergence pattern formula, Gaussian Incremental Loss Ratio, Hertig's Lognormal or Over-Dispersed Poisson models. In these models the one-year risk is lower than the ultimate risk at all sufficiently high confidence levels, see Theorems 2.1, 3.2, 3.4, 3.6. Finally, points 2 and 3 from Theorem 4.1 indicate that the distributions of the one-year risk and the ultimate risk have the same tail behaviour. This property holds in Gaussian Incremental Loss Ratio, Hertig's Lognormal and Over-Dispersed Poisson models where the distribution of X_n is uniquely specified by the reserve risk model, as well as if we model the one-year risk with the emergence pattern formula (2.1).

Let us remark that, instead of (4.1), we could also use the formula:

$$BE_1 = \tilde{\alpha}^2 X_n + (1-\tilde{\alpha}^2)\tilde{\mu} + \sqrt{\tilde{\alpha}^2(1-\tilde{\alpha}^2)}\tilde{\sigma}\xi, \tag{4.2}$$

with parameters $(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\alpha})$. We require that $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ and $Var[BE_1] = \alpha^2 Var[X_n]$, which imply that we have to choose $(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\alpha})$ such that

$$\begin{cases} \tilde{\alpha}^2\mu_{X_n} + (1-\tilde{\alpha}^2)\tilde{\mu} = \mu_{X_n}, \\ \tilde{\alpha}^4\sigma^2_{X_n} + \tilde{\alpha}^2(1-\tilde{\alpha}^2)\tilde{\sigma}^2 = \alpha^2\sigma^2_{X_n}. \end{cases} \tag{4.3}$$

We can observe that there are multiple solutions $(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\alpha})$ to (4.3). However, once $\tilde{\sigma}^2$ is set, there exists a unique solution $(\tilde{\mu}, \tilde{\alpha})$ to (4.3). Recalling the representation of the emergence pattern of the ultimate loss in the Gaussian Incremental Loss Ratio model, see (3.7) and (4.1), it is clear that $\tilde{\sigma}^2 = \sigma^2_{X_n}$ is the most obvious choice. Hence, we postulate to use the emergence pattern formula (4.1). We point out that if a different $\tilde{\sigma}^2$ is used in (4.2), than all conclusions concerning the one-year risk derived in this section still hold.

Example 4.1. First, we assume that $X_n \sim Exp(\lambda)$ with $1 - F_{X_n}(x) = e^{-\lambda x}$. In this case, the claims development process for the pair (BE_1, X_n) with the ultimate loss $X_n \sim Exp(\lambda)$ and the emergence pattern of the ultimate loss given by the conditional distribution (3.7) can be fully characterised, see Szatkowski (2019). From the assumed distributions of X_n and $BE_1|X_n$, we can derive the distributions:

$$\begin{aligned} Pr(BE_1 \le x) = &\ \Phi\Big(\frac{\lambda x - (1-\alpha^2)}{\sqrt{\alpha^2(1-\alpha^2)}}\Big) - exp\Big(\frac{3(1-\alpha^2) - 2\lambda x}{2\alpha^2}\Big)\Phi\Big(\frac{\lambda x - 2(1-\alpha^2)}{\sqrt{\alpha^2(1-\alpha^2)}}\Big), \\ X_n|BE_1 = x \sim &\ N_0\Big(\frac{\lambda x - 2(1-\alpha^2)}{\alpha^2\lambda}, \frac{1-\alpha^2}{\alpha^2\lambda^2}\Big), \end{aligned}$$

where N_0 denotes normal distribution left truncated in 0 and Φ denotes the standard normal distribution function. Moreover, we can show that $VaR_\gamma\big[BE_1 - \mathbb{E}[BE_1]\big] \sim \alpha^2 VaR_\gamma\big[X_n - \mathbb{E}[BE_1]\big]$ for $\gamma \to 1$.

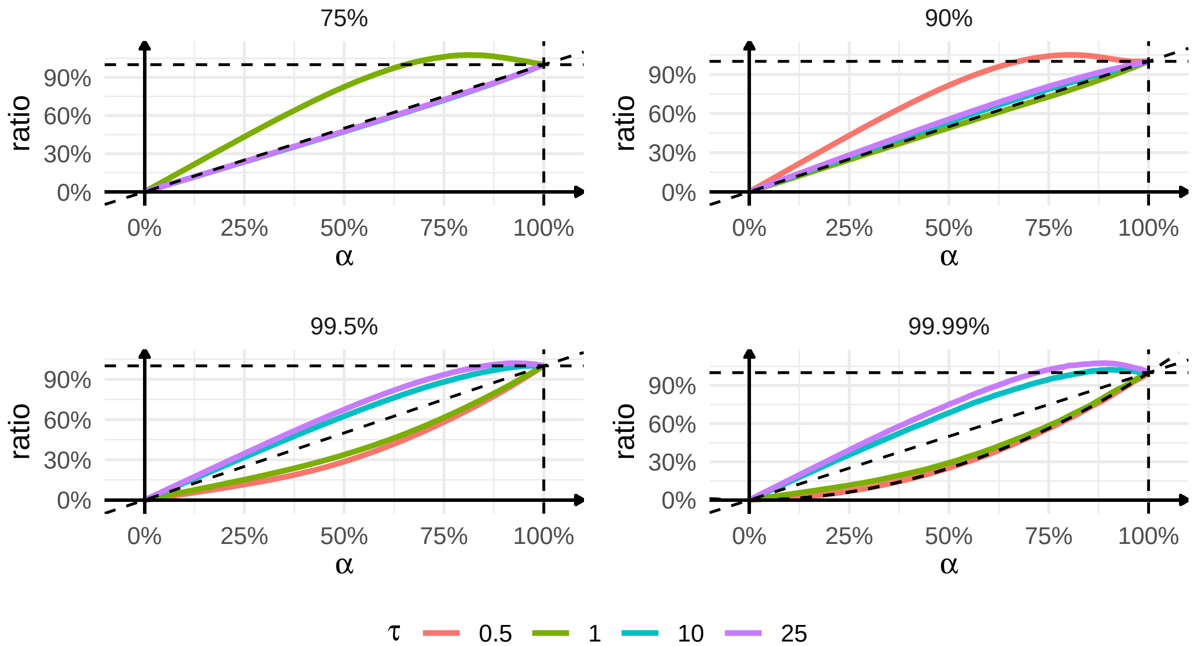

Figure 4: The ratios $\frac{VaR_\gamma[BE_1-\mathbb{E}[BE_1]]}{VaR_\gamma[X_n-\mathbb{E}[X_n]]}$ in the model where $X_n \sim Weibull(\tau)$ and $BE_1|X_n$ comes from the Gaussian ILR model.

To have more flexibility in our numerical study, we consider $X_n \sim Weibull(\lambda, \tau)$ with $1 - F_{X_n}(x) = e^{-\lambda x^\tau}$. We choose $\lambda = 1$ since λ is scaling parameter for the distribution of X_n. If $\tau = 1$, then $X_n \sim Exp(1)$. If $\tau > 2$, then $\lim_{x\to\infty}(1 - F_{X_n}(x))e^{vx^2} = 0$ for any $v > 0$. In order to calculate $VaR_\gamma[BE_1]$, we use Monte Carlo simulations and we generate a sample of size 10^6. The results are presented in Figure 4. For $\tau < 2$, the one-year risk is lower than the ultimate risk at sufficiently high confidence levels but the one-year risk is higher than the ultimate risk at low confidence levels. For $X_n \sim Exp(1)$, we confirm that $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] \approx \alpha^2 VaR_\gamma[X_n - \mathbb{E}[X_n]]$ for γ in the range $99.99\% - 99.999\%$ in our numerical experiments. For $\tau > 2$, we can observe the new property: the one-year risk is higher than the ultimate risk at all high confidence levels, in our case for $\gamma = 99.5\%$ and 99.99%. This implies that $VaR_\gamma\big[BE_1 - \mathbb{E}[BE_1]\big] \gtrsim VaR_\gamma\big[X_n - \mathbb{E}[BE_1]\big]$ for $\gamma \to 1$. The higher τ, the lighter the distribution of the ultimate loss and the higher the ratio $\frac{VaR_\gamma[BE_1-\mathbb{E}[BE_1]]}{VaR_\gamma[X_n-\mathbb{E}[X_n]]}$. □

We collect our new conclusions about the one-year risk and its relation with the ultimate risk.

Corollary 4.1. *Let us consider the model and the assumptions from Theorem 4.1.*

1. *For distributions of the ultimate loss which have right tail lighter than Gaussian distribution, the one-year risk is higher than the ultimate risk at all high confidence levels,*

2. *The limit* $\lim_{\gamma\to 1}\frac{VaR_\gamma[BE_1-E[BE_1]]}{VaR_\gamma[X_n-E[X_n]]}$ *can be different from* α *and depends on the distributions of* X_n *and* $BE_1|X_n$*,*

3. *The distributions of the one-year risk and the ultimate risk have the same tail behaviour.*

The property that the true one-year risk can be higher than the ultimate risk at all high confidence levels will be more visible in the next section where we show that the distributions of the one-year risk and the ultimate risk can have different tail behaviour.

4.2 The conditional distribution from Hertig's Lognormal model

From (3.13) we can conclude that the unconditional distribution of the best estimate of the ultimate loss in the model (3.10) is a mixture of log-normal distributions with an uncertain scaling factor. Consequently, we have the representation:

$$BE_1 = (X_n)^{\tilde{\alpha}^2} e^{(1-\tilde{\alpha}^2)\left(\tilde{m}+\frac{\tilde{\sigma}^2}{2}\right)+\sqrt{\tilde{\alpha}^2(1-\tilde{\alpha}^2)}\tilde{\sigma}\xi}, \tag{4.4}$$

where $\xi \sim N(0,1)$, and ξ is independent of X_n. The formula (4.4) is our new emergence pattern formula which allocates the ultimate loss with an arbitrary distribution to the best estimate of the ultimate loss. We can call (4.4) *the multiplicative lognormal emergence pattern formula*. The multiplicative lognormal emergence pattern can be seen as a version of the classical emergence pattern formula (2.1) where we allocate X_n to BE_1 with a random scaling factor in order to have a non-degenerate distribution of $BE_1|X_n = x$. Let us remark that we cannot simply use $BE_1 = UX_n$ where U is independent of X_n, since we would end up with $SD[BE_1] > SD[X_n]$ if $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$.

Theorem 4.2. *Let us assume that the ultimate loss is modelled with* $X_n \sim F_{X_n}$ *supported on* $(0,\infty)$*. Let* $\mu_{X_n} = E[X_n]$, $\psi_{X_n} = \frac{SD[X_n]}{E[X_n]}$ *and choose an emergence factor* $\alpha \in (0,1)$*. We assume that there exists a unique solution* $\tilde{\alpha} \in (0,1)$ *to the equation*

$$\frac{\mathbb{E}[X_n^{2\tilde{\alpha}^2}]}{(\mathbb{E}[X_n^{\tilde{\alpha}^2}])^2}\left(1+\psi_{X_n}^2\right)^{\tilde{\alpha}^2(1-\tilde{\alpha}^2)} = 1+\alpha^2\psi_{X_n}^2. \tag{4.5}$$

We set

$$\begin{aligned}\tilde{m} &= \frac{\log(\mu_{X_n}) - \log\left(\mathbb{E}[X_n^{\tilde{\alpha}^2}]\right) - (1-\tilde{\alpha}^4)\frac{\tilde{\sigma}^2}{2}}{1-\tilde{\alpha}^2},\\ \tilde{\sigma}^2 &= \log\left(1+\psi_{X_n}^2\right).\end{aligned}$$

The emergence pattern of the ultimate loss X_n *is described with the formula* (4.4)*. We have the following properties:*

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$*,*

2. *If X_n has a light-tailed distribution, then BE_1 has a subexponential distribution with all moments finite. Moreover, $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] > VaR_\gamma[X_n - \mathbb{E}[X_n]]$ for $\gamma > \gamma^*$, with some $\gamma^* < 1$,*

3. *If X_n has a subexponential distribution with all moments finite such that $\limsup_{x\to\infty} \frac{1-F_{X_n}(vx)}{1-F_{X_n}(x)} < 1$ for some $v > 1$ and $X_n^{\tilde{\alpha}^2}$ is light-tailed or subexponential, then BE_1 has a subexponential distribution with all moments finite,*

4. *If X_n has a Pareto-type distribution with tail index θ and regularly varying function L such that $L(x^{\frac{1}{\tilde{\alpha}^2}}) \sim C \cdot L(x)$, $x \to \infty$, then BE_1 has a Pareto-type distribution with tail index $\theta/\tilde{\alpha}^2 > \theta$, and*

$$\lim_{\gamma\to 1} \frac{VaR_\gamma\big[BE_1 - E[BE_1]\big]}{VaR_\gamma\big[X_n - E[X_n]\big]} = 0.$$

Remark 4.1. *Point 3 is satisfied e.g. for lognormal, Weibull, Benktander type I and type II distributions, see e.g. the remark before Theorem 2.1 in Tang (2006). Point 4 is satisfied e.g. if $L(x) \sim a(\log(x))^b$, $x \to \infty$, which is the case for Pareto, Burr and loggamma distributions.*

Let us investigate the equation (4.5).

Lemma 4.1. *Let us consider the equation* (4.5) *with $\alpha \in (0, 1)$.*

1. *There exists at least one solution $\tilde{\alpha} \in (0, 1)$ to* (4.5)*,*

2. *There exists a unique solution $\tilde{\alpha} \in (0, 1)$ to* (4.5) *for Gamma, Weibull, Pareto (for shape parameter greater than* 2.15*) and loggamma (for shape parameter greater than* 2.15*) distributions of X_n.*

Remark 4.2. *For Pareto and loggamma distributions with shape parameters close to 2, the equation* (4.5) *may have several solutions. Low values of shape parameter are unlikely to be used in practice in our model since the emergence factor α is based on variance and variance of Pareto or loggamma explodes when the shape parameter approaches* 2*.*

We have found another interesting property of the one-year risk vs. the ultimate risk. Theorem 4.2 shows that, in general, the distribution of the one-year risk can have a different tail behaviour than the distribution of the ultimate risk. If the tail of the one-year risk is heavier than the tail of the ultimate risk, as in point 2 from Theorem 4.2, then we can expect that the one-year risk is higher than the ultimate risk at all high confidence levels, and we have already observed such a property in the previous section in a different claims development model. If the tail of the one-year risk is lighter than the tail of the ultimate risk, as in point 4 from Theorem 4.2, then we can expect that the one-year risk is substantially lower than the ultimate risk at all high confidence levels.

As discussed in the previous section, we can choose different parameters $(\tilde{m}, \tilde{\sigma}^2, \tilde{\alpha})$ in (4.4) and guarantee that $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ and $Var[BE_1] = \alpha^2 Var[X_n]$, see the proof of Theorem 4.2. Recalling the representation of the emergence pattern of the ultimate loss in the Hertig's Lognormal model, see (3.13) and (4.4), it is clear that $\tilde{\sigma}^2 = \log(1 + \psi^2_{X_n})$ is the most obvious choice. We point out that if a different $\tilde{\sigma}^2$ is used in (4.4), than all conclusions concerning the one-year risk derived in this section still hold.

Example 4.3. We again consider $X_n \sim Weibull(\tau)$. In order to calculate $VaR_\gamma[BE_1]$, we use Monte Carlo simulations and we generate a sample of size 10^6. The results are presented in Figure 5. For $\tau = 10$ and $\alpha = 85\%$, the one-year risk is higher than the ultimate risk at $\gamma = 99.5\%$ by about 0.5%, and at $\gamma = 99.99\%$ by about 10%. If the emergence pattern formula (2.1) is applied in this claims development model to measure the one-year risk, then the true one-year risk is underestimated, respectively, by about 15% and 23% at $\gamma = 99.5\%$ and 99.99%. For $\alpha = 85\%$, the one-year risk is higher than the ultimate risk at the confidence level $\gamma = 99.5\%$, and at all confidence levels $\gamma > 99.5\%$, if we consider Weibull distributions with $\tau > 7.5$.

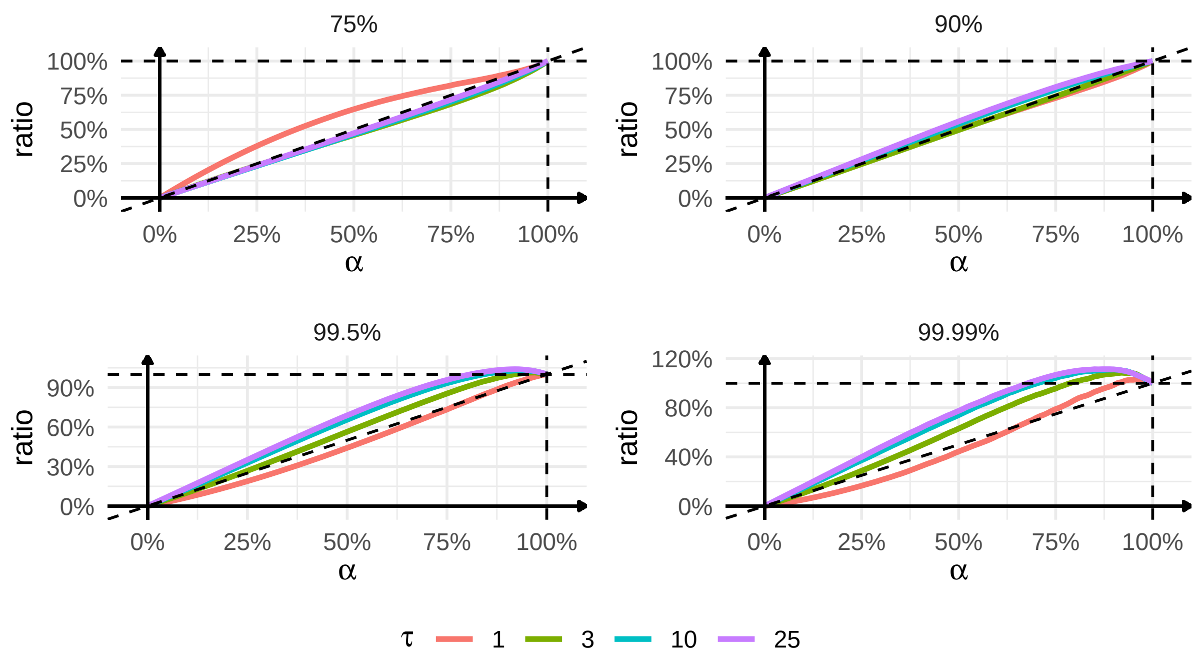

Figure 5: The ratios $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the model where $X_n \sim Weibull(\tau)$ and $BE_1|X_n$ comes from the Hertig's Lognormal model.

We also consider $X_n \sim Pareto(1, \theta)$ with $1 - F_{X_n}(x) = x^{-\theta}, x > 1$. Then, $\log(X_n)$ has

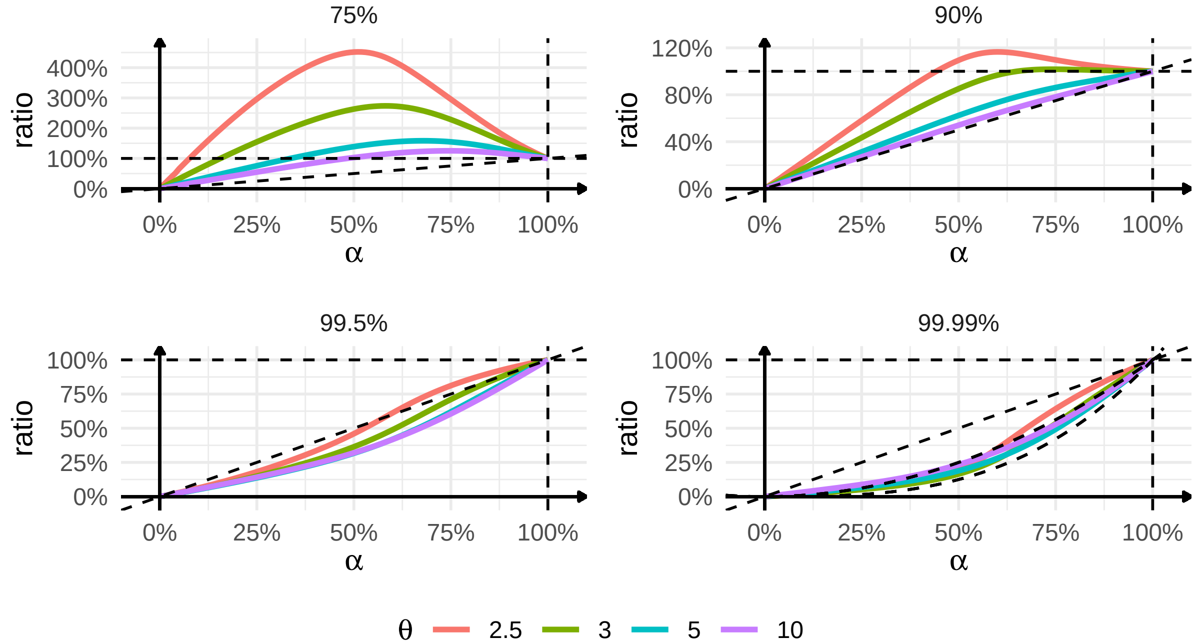

Figure 6: The ratios $\frac{VaR_\gamma[BE_1-\mathbb{E}[BE_1]]}{VaR_\gamma[X_n-\mathbb{E}[X_n]]}$ in the model where $X_n \sim Pareto(\theta)$ and $BE_1|X_n$ comes from the Hertig's Lognormal model.

exponential distribution with parameter θ and we can use the results from Example 4.1 to calculate $VaR_\gamma[BE_1]$, see Szatkowski (2019). The results are presented in Figure 6. For $\theta = 5$ and $\alpha = 50\%$, the one-year risk is lower than the ultimate risk at $\gamma = 99.5\%$ by about 68% and at $\gamma = 99.99\%$ by about 81%. If the emergence pattern formula (2.1) is applied in this claims development model to measure the one-year risk, then the true one-year risk is overestimated, respectively, by about 58% and 165% at $\gamma = 99.5\%$ and 99.99%. We can also observe that the one-year risk can be significantly higher than the ultimate risk at low confidence levels. Finally, even though point 4 from Theorem 4.2 shows that $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] = o\big(VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]\big)$ for $\gamma \to 1$, we end up with $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] \approx \alpha^3 VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]$ for γ in the range $99.99\% - 99.999\%$ in our numerical experiments. □

Corollary 4.2. *Let us consider the model and the assumptions from Theorem 4.2.*

1. *The distributions of the one-year risk and the ultimate risk can have different tail behaviour. For light-tailed distributions of the ultimate loss, the one-year risk has heavier tails than the ultimate risk. For Pareto-type distributions of the ultimate loss, the one-year risk has lighter tails than the ultimate risk,*

2. *For light-tailed distributions of the ultimate loss, the one-year risk is higher than the ultimate risk at all high confidence levels,*

3. *For Pareto-type distributions of the ultimate loss, the one-year risk vanishes compared to the ultimate risk for the confidence level in the limit $\gamma \to 1$. In practical examples, the ratio between the true one-year risk and the ultimate risk at very high confidence levels can be of order α^3.*

Point 1 from Corollary 4.1 and point 2 from Corollary 4.2 show that the common believe among actuaries that the one-year risk is always lower than the ultimate risk is not true. In Solvency II we have to use the confidence level of 99.5% or higher. In many claims development models, including the most common reserve risk models presented in Section 3, if a sufficiently high confidence level for Value-at-Risk is chosen, then the one-year risk is lower than the ultimate risk, which agrees with the common believe. However, there are claims development models, and examples of such model are discussed in Sections 4.1-4.2, where the one-year risk is higher than the ultimate risk at all sufficiently high confidence levels. Actuaries should be aware of this property in order not to misunderstand and underestimate the one-year risk. On the other side, the one-year risk can be much less dangerous than it is believed by users of the linear emergence pattern formula (2.1). Point 3 from Corollary 4.2 proves that, in a claims development model with the emergence pattern of the ultimate loss given by (4.4), the tail index of the Pareto-type distribution of the true one-year risk is higher than the tail index of the Pareto-type distribution of the ultimate risk, whereas from Theorem 2.1 we know that the tail index of the Pareto-type distribution of the one-year risk predicted with (2.1) remains equal to the tail index of the distribution of the ultimate loss. The comparison of the tail indices can be found in Table 1.

Table 1: Tail indices of BE_1 if $X_n \sim Pareto(1, \theta)$ and $BE_1|X_n$ comes from the Hertig's Lognormal model.

$\theta/\tilde{\alpha}^2$	$\alpha = 15\%$	$\alpha = 50\%$	$\alpha = 85\%$
$\theta = 2.5$	80.5	6.1	2.7
$\theta = 5$	213.4	17.9	6.3
$\theta = 10$	439.8	38.2	13.2

5 Proofs

We denote $\mu = \mu_{X_n}$. By $f(x) \sim Cg(x)$ we mean the limit $x \to \infty$, unless a different limit is specified.

The proof of Theorem 2.1. Points 1 and 4. The results are obvious.

Point 2. The result for light-tailed distributions is obvious, since a distribution is light-tailed i.f.f it has a moment generating function. We consider a subexponential distribution with all moments finite. From the definition of subexponential distribution, $Pr(X_1 + ... + X_n > x) \sim nPr(X > x)$, we deduce that if X_n is subexponential than αX_n is also subexponential. Since $BE_1^{ep} = \alpha X_n + (1-\alpha)\mu$ is a convolution of a subexponential distribution with a degenerate distribution, then BE_1^{ep} is subexponential by Lemma 2.4.4.a in Mikosch (1999) and Lemma 9 in Geluk, De Vries (2006). Since X_n has all moments finite, then BE_1 has all moments finite as well.

Point 3. Let L denote a slowly varying function which characterises the distribution of X_n, i.e. $1-F_{X_n}(x) = L(x)x^{-\theta}$. By the Karamata representation of L, see e.g. Corollary 2.1 in Resnick (2007), we know that $L(x) = c(x)e^{\int_1^x t^{-1}d(t)dt}, x > 0$, for some functions $c(x) \to c, d(x) \to 0$, as $x \to \infty$. We can deduce that $L(ax+b) \sim L(x)$ for any $a \geq 1$ and $b \in \mathbb{R}$. We can prove the following asymptotic relations:

$$\begin{aligned}
1 - F_{X_n-\mu}(x) &= L(x+\mu)(x+\mu)^{-\theta} \sim L(x)x^{-\theta} = 1 - F_{X_n}(x),\\
1 - F_{BE_1^{ep}-\mu}(x) &= 1 - F_{\alpha(X_n-\mu)}(x) = 1 - F_{X_n}\left(\frac{x}{\alpha}+\mu\right) = L\left(\frac{x}{\alpha}+\mu\right)\left(\frac{x}{\alpha}+\mu\right)^{-\theta}\\
&\sim \alpha^\theta L(x)x^{-\theta} = \alpha^\theta(1 - F_{X_n}(x)) \sim \alpha^\theta(1 - F_{X_n-\mu}(x)),\\
1 - F_{BE_1^{ep}}(x) &= 1 - F_{\alpha X_n + (1-\alpha)\mu}(x) \sim \alpha^\theta L(x)x^{-\theta}.
\end{aligned} \tag{5.1}$$

The distribution of BE_1 can be derived from the last relation in (5.1). □

The proofs of Theorems 3.1-3.6 and Proposition 3.1. We use standard properties of normal, lognormal, Poisson distributions, their moments, quantiles and conditional distributions. The parameterisations of the distributions from Theorems 3.1, 3.3, 3.5 can be checked by direct substitution, once the distributions are derived by classical methods. The complete derivations are presented in Szatkowski (2019).

Points 2-3 from Theorem 3.4. We use the distribution (3.14) and the formulas

$$VaR_\gamma[X_n] = e^{\tilde{m}+\tilde{\sigma}\Phi^{-1}(\gamma)}, \quad VaR_\gamma[BE_1] = e^{\tilde{m}+(1-\tilde{\alpha}^2)\frac{\tilde{\sigma}^2}{2}+\tilde{\alpha}\tilde{\sigma}\Phi^{-1}(\gamma)}, \tag{5.2}$$

which describe the quantiles of the lognormal distributions of BE_1 and X_n. We can solve the inequality $VaR_\gamma[BE_1] < \alpha^k VaR_\gamma[X_n]$, for $k \in \mathbb{N}$, which implies that $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] < \alpha^k VaR_\gamma[X_n - \mathbb{E}[X_n]]$. Since $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$, $VaR_\gamma[BE_1] > VaR_\gamma[X_n]$ is equivalent to $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] > VaR_\gamma[X_n - \mathbb{E}[X_n]]$. The limit $\lim_{\gamma\to 1} \frac{VaR_\gamma[BE_1-\mathbb{E}[BE_1]]}{VaR_\gamma[X_n-\mathbb{E}[X_n]]}$ can be directly calculated using the formula (5.2).

Point 2 from Theorem 3.6. Let $Y \sim Poiss(\alpha^2\lambda)$ and $X \sim Poiss(\lambda)$. It is known that $F_X(x) \leq F_Y(x)$ for all x, see e.g. the remark at the end of section 1.2 in Klenke and Mattner (2010). Let us choose arbitrary $k > 0$. Our first goal is to show that $F_X(x+k) < F_Y(x)$ for all sufficiently high x. We only consider $x > \lambda$. By Theorem 2 from Short (2013), we have the bounds:

$$\Phi\Big(\sqrt{2H(\lambda, x)}\Big) < F_X(x) < \Phi\Big(\sqrt{2H(\lambda, x+1)}\Big),$$

where $H(\lambda, x) = \lambda - x + x\log\left(\frac{x}{\lambda}\right)$. Consequently, our assertion holds if $H(\alpha^2\lambda, x) > H(\lambda, x+k+1)$ for all sufficiently high x. Using the definition of H, we have to investigate the inequality:

$$\begin{aligned} x\log\left(\frac{x}{x+k+1}\right) &+ k + 1 \\ &-(k+1)\log(x+k+1) - x\log\left(\alpha^2\right) > \lambda(1-\alpha^2) - (k+1)\log\lambda. \end{aligned} \tag{5.3}$$

We can show that the first two terms on the left-hand side of (5.3) converge to zero as $x \to +\infty$, whereas the last two terms of the left-hand side of (5.3) converge to $+\infty$ as $x \to +\infty$. Hence, the inequality (5.3) is satisfied for all sufficiently high x.

Let us consider our claims development model. For simplicity we assume that $\phi_{X_n} = 1$. Let $Y = BE_1 - (1-\alpha^2)\lambda$. By the arguments from above, we know that $F_Y(x) > F_{X_n}\big(x + (1-\alpha^2)\lambda\big)$ for all sufficiently high x. Since the distribution F_{BE_1} is the distribution F_Y shifted to the right by $(1-\alpha^2)\lambda$, we deduce that $F_{BE_1}(x) > F_{X_n}(x)$ for all sufficiently high x. The result is proved.

Point 3 from Theorem 3.6. By Lemma 9.1 from Steutel and Van Harn (2004), if $X \sim Poiss(\lambda)$, then $-\log(1-F_X(x)) \sim x\log(x)$. Consequently, we can derive the limit:

$$\lim_{t\to\infty} \frac{\log(1-F_{BE_1}(tx))}{\log(1-F_{X_n}(t))} = x, \quad x > 0. \tag{5.4}$$

We use the arguments from Propositions 2.2 and 2.6.vi from Resnick (2007). Since the limit (5.4) holds for a sequence of functions, the sequence of inverse functions also converges. We can establish the limit:

$$\lim_{t\to\infty} \frac{F_{BE_1}^{-1}\Big(1-(1-F_{X_n}(t))^y\Big)}{t} = y, \quad y > 0. \tag{5.5}$$

By the property that $F_{X_n}(F_{X_n}^{-1}(t)) \sim t, t \to 1$, we end up with

$$\lim_{t\to\infty} \frac{F_{BE_1}^{-1}(1-1/t^y)}{F_{X_n}^{-1}(1-1/t)} = y, \quad y > 0. \tag{5.6}$$

Finally, we choose $y = 1$ in (5.6). We conclude that $VaR_\gamma[BE_1 - \mathbb{E}[BE_1]] \sim VaR_\gamma[X_n - \mathbb{E}[X_n]]$ for $\gamma \to 1$. □

The proof Theorem 4.1. Point 1 The result is obvious.

Point 2. The result for light-tailed distributions is obvious. We consider a subexponential distribution with all moments finite. By the proof of Theorem 2.1, we know that $Z_1 = \alpha^2 X_n + (1-\alpha^2)\mu_{X_n}$ is subexponential with all moments finite. The random variable $Z_2 = \sqrt{\alpha^2(1-\alpha^2)}\sigma_{X_n}\xi$ has normal distribution and is light-tailed. We observe that

$$\lim_{x\to\infty} \frac{1-F_{Z_2}(x)}{1-F_{Z_1}(x)} = \lim_{x\to\infty} \frac{e^{tx}(1-F_{Z_2}(x))}{e^{tx}(1-F_{Z_1}(x))} = 0, \quad for\ some\ t>0, \tag{5.7}$$

since the numerator converges to zero (Z_2 is light-tailed) and the denominator converges to infinity (Z_1 is subexponential). Consequently, $1-F_{Z_2}(x) = o\big(1-F_{Z_1}(x)\big)$. By Lemma 2.4.4.a in Mikosch (1999) and Lemma 9 in Geluk, De Vries (2006), $BE_1 = Z_1 + Z_2$ has a subexponential distribution. It is clear that BE_1 has all moments finite, since Z_1 and Z_2 have all moments finite.

We prove the second statement. We can derive the lower bound for the convolution:

$$\begin{aligned} 1-F_{BE_1}(x) &= Pr\big(\alpha^2 X_n + (1-\alpha^2)\mu_{X_n} + Z_2 > x\big) \geq Pr\big((1-\alpha^2)\mu_{X_n} + Z_2 > x, \alpha^2 X_n < x\big) \\ &= Pr\big((1-\alpha^2)\mu_{X_n} + Z_2 > x\big) F_{X_n}\Big(\frac{x}{\alpha^2}\Big), \quad x>0. \end{aligned}$$

Using a bound for the tail of normal distribution, see Gordon (1941), the assumption that $\lim_{x\to\infty}(1-F_{X_n}(x))e^{vx^2} = 0$ and $\lim_{x\to\infty}\frac{e^{wx}}{x} = +\infty$ for all $w>0, v>0$, we can show that

$$\begin{aligned} \lim_{x\to\infty}\frac{1-F_{BE_1}(x)}{1-F_{X_n}(x)} &\geq \lim_{x\to\infty}\frac{Pr\big((1-\alpha^2)\mu_{X_n}+Z_2>x\big)F_{X_n}(\frac{x}{\alpha^2})}{1-F_{X_n}(x)} \\ &\geq \frac{1}{\sqrt{2\pi}}\lim_{x\to\infty}\frac{e^{-\frac{(u(x))^2}{2}}F_{X_n}(\frac{x}{\alpha^2})}{(u(x)+\frac{1}{u(x)})(1-F_{X_n}(x))} \\ &\geq K\lim_{x\to\infty}\frac{e^{wx}}{e^{vx^2}(u(x)+\frac{1}{u(x)})(1-F_{X_n}(x))} = +\infty, \end{aligned}$$

where $u(x) = \frac{x-(1-\alpha^2)\mu_{X_n}}{\sqrt{\alpha^2(1-\alpha^2)\sigma^2_{X_n}}}$ and K, v, w denote some positive constants. We can deduce that $1-F_{BE_1}(x) > 1-F_{X_n}(x)$ for all sufficiently high x, and $VaR_\gamma[BE_1] > VaR_\gamma[X_n]$ for all sufficiently high γ.

Point 3. By similar arguments as in (5.1) and (5.7), $1-F_{Z_1}(x) \sim \alpha^{2\theta}\big(1-F_{X_n}(x)\big)$ and $1-F_{Z_2}(x) = o\big(1-F_{Z_1}(x)\big)$. By Lemma 2.4.4.a in Mikosch (1999) and Lemma 9 in Geluk, De Vries (2006), we derive that

$$1-F_{BE_1}(x) \sim 1-F_{Z_1}(x) \sim \alpha^{2\theta}\big(1-F_{X_n}(x)\big),$$

and we deduce that BE_1 inherits the tail from X_n. Using (5.1) again, we prove that $1-F_{BE_1-\mu}(x) \sim \alpha^{2\theta}(1-F_{X_n-\mu}(x))$. The result for VaR follows from the arguments used in Proposition 2.6.vi from Resnick (2007). □

The proof Theorem 4.2. Points 1 and 2. Using independence between X_n and $\xi \sim N(0,1)$, as well as moments for log-normal distribution, we can calculate the moments of BE_1:

$$\mathbb{E}\big[BE_1^k\big] = \mathbb{E}\Big[(X_n)^{k\tilde{\alpha}^2}\Big]e^{k(1-\tilde{\alpha}^2)\left(\tilde{m}+\frac{\tilde{\sigma}^2}{2}\right)+\frac{1}{2}k^2\tilde{\alpha}^2(1-\tilde{\alpha}^2)\tilde{\sigma^2}}, \quad k=1,2. \tag{5.8}$$

We require that $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ and $Var[BE_1] = \alpha^2 Var[X_n]$. If we use (5.8), then we derive the equation for the parameters $(\tilde{\sigma}^2, \tilde{\alpha})$:

$$\frac{\mathbb{E}\big[(X_n)^{2\tilde{\alpha}^2}\big]}{\big(\mathbb{E}\big[(X_n)^{\tilde{\alpha}^2}\big]\big)^2}e^{\tilde{\alpha}^2(1-\tilde{\alpha}^2)\tilde{\sigma^2}} = 1+\alpha^2\psi^2_{X_n}. \tag{5.9}$$

We can see that we can choose any $\tilde{\sigma}^2 > 0$ and set $1+\Psi^2_{X_n} = e^{\tilde{\sigma^2}}$ in (5.9).

Point 2. Since X_n has a moment generating function and $\tilde{\alpha} \in (0,1)$, then $Z_1 = X_n^{\tilde{\alpha}^2}$ has a moment generating function and has a light-tailed distribution. The lognormal distribution $Z_2 = e^{(1-\tilde{\alpha}^2)\left(\tilde{m}+\frac{\tilde{\sigma}^2}{2}\right)+\sqrt{\tilde{\alpha}^2(1-\tilde{\alpha}^2)}\tilde{\sigma}\xi}$ is subexponential and satisfies $\limsup_{x\to\infty}\frac{1-F_{Z_2}(vx)}{1-F_{Z_2}(x)} < 1$ for some $v > 1$, see the remark before Theorem 2.1 in Tang (2006). Moreover, we can observe that

$$\lim_{x\to\infty}\frac{1-F_{Z_1}(vx)}{1-F_{Z_2}(x)} = \lim_{x\to\infty}\frac{e^{tvx}(1-F_{Z_1}(vx))}{e^{t(v-1)x}e^{tx}(1-F_{Z_2}(x))} = 0, \quad for\ some\ t>0,\ for\ all\ v \geq 1, \tag{5.10}$$

since the numerator converges to zero and the denominator converges to infinity. Consequently, $1-F_{Z_1}(vx) = o\big(1-F_{Z_2}(x)\big)$ for all $v \geq 1$. From Theorem 2.1 and Corollary 2.1.1 in Tang (2006), we can conclude that $BE_1 = Z_1Z_2$ is subexponential. By (5.8), BE_1 has all moments finite. Finally, we prove that

$$\lim_{x\to\infty}\frac{1-F_{X_n}(x)}{1-F_{BE_1}(x)} = \lim_{x\to\infty}\frac{e^{tx}(1-F_{X_n}(x))}{e^{tx}(1-F_{BE_1}(x))} = 0, \quad for\ some\ t>0,$$

since the numerator converges to zero and the denominator converges to infinity. We can deduce that $1-F_{BE_1}(x) > 1-F_{X_n}(x)$ for all sufficiently high x, and $VaR_\gamma[BE_1] > VaR_\gamma[X_n]$ for all sufficiently high γ.

Point 3. If the distribution of X_n satisfies the assumption that $\limsup_{x\to\infty}\frac{1-F_{X_n}(vx)}{1-F_{X_n}(x)} < 1$ for some $v > 1$, then the distribution of $X_n^{\tilde{\alpha}^2}$, given by $F_{X_n^{\tilde{\alpha}}}(x) = F_{X_n}(x^{\frac{1}{\tilde{\alpha}}})$, also satisfies this assumption. If $X_n^{\tilde{\alpha}}$ has a light-tailed distribution, then the result follows from point 2. If $X_n^{\tilde{\alpha}}$ is subexponential, then the result follows from Theorem 2.1 and Corollary 2.1.2 in Tang (2006). Indeed, we can show that

$$\begin{aligned}\lim_{x\to\infty}\frac{1-F_{Z_2}(vx)}{1-F_{Z_2}(x)} &\leq \lim_{x\to\infty}\frac{(\log(x)-\mu)^2+\sigma^2}{(\log(vx)-\mu)(\log(x)-\mu)}\frac{e^{-\frac{(\log(vx)-\mu)^2}{2\sigma^2}}}{e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}}}\\ &= \lim_{x\to\infty}\frac{(\log(x)-\mu)^2+\sigma^2}{(\log(vx)-\mu)(\log(x)-\mu)}e^{\frac{-\log(v)}{2\sigma^2}(2(\log(x)-\mu)+\log(v))} = 0, \quad for \quad all \quad v>1,\end{aligned}$$

where μ and σ denote the parameters of the log-normal distribution Z_2, and we use bounds for the tail of normal distribution, see Gordon (1941). By (5.8), BE_1 has all moments finite since X_n has all moments finite.

Point 4. We can immediately prove the relation:

$$1 - F_{X_n^{\tilde{\alpha}^2}}(x) = L(x^{\frac{1}{\tilde{\alpha}^2}})x^{-\frac{\theta}{\tilde{\alpha}^2}} \sim CL(x)x^{-\frac{\theta}{\tilde{\alpha}^2}}.$$

From Proposition 1.3.9 from Mikosch (1999), we derive the asymptotic formula:

$$1 - F_{BE_1}(x) \sim C\mathbb{E}\big[Z_2^{\frac{\theta}{\tilde{\alpha}^2}}\big]L(x)x^{-\frac{\theta}{\tilde{\alpha}^2}}. \tag{5.11}$$

Consequently, BE_1 has the tail index equal to $\frac{\theta}{\tilde{\alpha}^2}$. Since X_n has a Pareto-type distribution $1 - F(x) = L(x)x^{-\theta}$, we have the asymptotic formula:

$$VaR_\gamma\big[X_n\big] \sim \big(1-\gamma\big)^{-1/\theta}\tilde{L}\Big(\frac{1}{1-\gamma}\Big), \quad \gamma \to 1, \tag{5.12}$$

where $\tilde{L}$ denotes the de Bruyn conjugate of L, see (2.7) and Proposition 2.5 from Berlaint et al. (2004). Using (5.11)-(5.12) and the arguments from Proposition 2.6.vi from Resnick (2007), we can establish that

$$\begin{aligned} VaR_\gamma[BE_1] &\sim \Big(C\mathbb{E}\big[Z_1^{\frac{\theta}{\tilde{\alpha}^2}}\big]\Big)^{\frac{\tilde{\alpha}^2}{\theta}}(1-\gamma)^{-\frac{\tilde{\alpha}^2}{\theta}}\tilde{L}\Big(\frac{1}{1-\gamma}\Big) \\ &\sim \Big(C\mathbb{E}\big[Z_1^{\frac{\theta}{\tilde{\alpha}^2}}\big]\Big)^{\frac{\tilde{\alpha}^2}{\theta}}(1-\gamma)^{\frac{1-\tilde{\alpha}^2}{\theta}}VaR_\gamma[X_n], \quad \gamma \to 1, \end{aligned}$$

and the result is proved. □

The proof of Lemma 4.1. Let $k = \tilde{\alpha}^2$. We simply denote $X = X_n, \psi = \psi_{X_n}$. We choose $\alpha \in (0,1)$. We introduce the functions:

$$f(k) = \mathbb{E}[X^k] = \mathbb{E}[e^{k\log(X)}], \quad h(k) = \log f(2k) - 2\log f(k) + k(1-k)\log(1+\psi^2).$$

It is easy to show that $h(k)$ is continuous on $(0,1)$. We look for $k \in (0,1)$ which satisfies the equation

$$h(k) = \log(1 + \alpha^2\psi^2). \tag{5.13}$$

We have $h(0) = 0$ and $h(1) = \log(1+\psi^2) > \log(1+\alpha^2\psi^2)$. It is known that $k \mapsto f(k)$ is log-convex. Hence $h'(k) = 2\big(\frac{d}{dx}\log f(x)|_{x=2k} - \frac{d}{dx}\log f(x)|_{x=k}\big) + (1-2k)\log(1+\psi^2) > 0$, and $k \mapsto h(k)$ is increasing on $(0, \frac{1}{2})$. We can conclude that there exists at least one solution to (5.13).

Gamma distribution. Let $X \sim Gamma(a,b)$. By straightforward calculations, we can show that

$$f(k) = \mathbb{E}[X^k] = b^k\frac{\Gamma(k+a)}{\Gamma(a)}, \quad \frac{d^2}{dk^2}\log f(k) = \frac{d^2}{dk^2}\log\Gamma(k+a),$$

and

$$\begin{aligned}\frac{d^2}{dk^2}h(k) &= 4\frac{d^2}{dx^2}\log\Gamma(x+a)|_{x=2k} - 2\frac{d^2}{dx^2}\log\Gamma(x+a)|_{x=k} - 2\log(1+\psi^2)\\ &= \frac{d^2}{dk^2}\log\Big(\frac{\Gamma(2k+a)}{\big(\Gamma(k+a)\big)^2}\Big) - 2\log(1+\psi^2) = \frac{d^2}{dk^2}g(k) - 2\log(1+\psi^2),\end{aligned}$$

where we define

$$g(k) = \log\Big(\frac{\Gamma(2k+a)}{\big(\Gamma(k+a)\big)^2}\Big).$$

Let $a \geq \frac{1}{2}$. By Theorem 5 from Qi et al. (2008), the function g is 3-log-concave, which means that $\frac{d^3}{dk^3}\log g(k) \leq 0$. We can conclude that the function $k \mapsto \frac{d^2}{dk^2}\log g(k)$ is decreasing. We have to consider three cases: 1) $\frac{d^2}{dk^2}g(k) > 2\log(1+\psi^2)$ for all $k \in (0,1)$: the function $k \mapsto h(k)$ is convex on $(0,1)$ which implies, together with the properties mentioned in the beginning, that there exists a unique solution to (5.13); 2) $\frac{d^2}{dk^2}g(k) < 2\log(1+\psi^2)$ for all $k \in (0,1)$: the function $k \mapsto h(k)$ is concave on $(0,1)$ which implies that there exists a unique solution to (5.13); 3) There exists $k_0 \in (0,1)$ such that $\frac{d^2}{dk^2}g(k) > 2\log(1+\psi^2)$ for $0 < k < k_0$ and $\frac{d^2}{dk^2}g(k) < 2\log(1+\psi^2)$ for $1 > k > k_0$: the function $k \mapsto h(k)$ is convex on $(0,k_0)$ and concave on $(k_0,1)$ which again implies that there exists a unique solution to (5.13).

Let $a < \frac{1}{2}$. From series representation of trigamma function, we deduce that the second derivative of loggamma function is positive. By inequality (1.10) from Guo et al. (2015), which gives upper and lower bounds on the second derivative of loggamma function, we get

$$\frac{d^2}{dk^2}g(k) \leq \frac{2a+3}{(2k+a+\frac{1}{2})(k+a+1)} + \frac{2a^2-4k^2}{(2k+a)^2(k+a)^2}. \tag{5.14}$$

Since $k \mapsto h(k)$ is increasing on $(0,\frac{1}{2})$, we only have to consider $k \in (\frac{1}{2},1)$. Using (5.14) and the assumption that $k \in (\frac{1}{2},1)$, we derive the inequality

$$\frac{d^2}{dk^2}h(k) = \frac{d^2}{dk^2}g(k) - 2\log\big(1+\frac{1}{a}\big) \leq 2\Big(\frac{1}{a+\frac{3}{2}} - \log\big(1+\frac{1}{a}\big) + \frac{a^2-2k^2}{(2k+a)^2(k+a)^2}\Big). \tag{5.15}$$

Let us introduce the function $p(a) = \frac{1}{a+\frac{3}{2}} - \log(1+\frac{1}{a})$. We have $p(0^+) = -\infty$ and $p(\frac{1}{2}) = \frac{1}{2} - \log 3 < 0$. Moreover, by straightforward calculation, we can check that $p'(a) > 0$ for $a \in (0,\frac{1}{2})$. Consequently, the first two terms on the right hand side of (5.15) are negative. It is clear that the last term in (5.15) is also negative. Hence, we have proved that the function $k \mapsto h(k)$ is concave on $(\frac{1}{2},1)$. This property implies, together with the properties mentioned in the beginning, that there exists a unique solution to (5.13).

Pareto distribution. Let $X \sim Pareto(x_0,\theta)$ with $1-F(x) = x_0^\theta x^{-\theta}$ for $x > x_0$. We assume that $\theta > 2$ so that $Var[X] < \infty$. By straightforward calculations, we can show that

$$f(k) = \mathbb{E}[X^k] = \frac{\theta x_0^k}{\theta - k}, \quad \frac{d}{dk}\log f(k) = \log(x_0) + \frac{1}{\theta-k}.$$

We can also calculate that

$$\frac{d}{dk}h(k) = \frac{2}{\theta - 2k} - \frac{2}{\theta - k} - (2k-1)\log(1+\psi^2).$$

Since $\psi^2 = \frac{1}{\theta(\theta-2)}$ and $\log(1+x) < x$, we can prove the inequality:

$$\frac{d}{dk}h(k) > \frac{\theta^2 + \theta(6k^2 - 7k) + 2k^2 - 4k^3}{(\theta-2)\theta(\theta-2k)(\theta-k)}. \tag{5.16}$$

The denominator of the fraction on the right hand side of (5.16) is always positive and the numerator is a quadratic function in θ. The larger root of this quadratic function is given by $\tilde{\theta}(k) = \frac{1}{2}k(\sqrt{36k^2 - 68k + 41} - 6k + 7)$. One can check that the maximal value of the function $k \mapsto \tilde{\theta}(k)$ on $(0,1)$ is equal to $\theta^* \approx 2.15$. Consequently, we can conclude that for $\theta > \theta^*$ the first derivative (5.16) is positive for any $k \in (0,1)$, meaning that the function $k \mapsto h(k)$ is increasing on $(0,1)$. This property implies, together with the properties mentioned in the beginning, that there exists a unique solution to (5.13).

Weibull distribution. The proof is analogous as for Gamma distribution.

Loggamma distribution. The proof is analogous as for Pareto distribution. □

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