# Fair valuation of insurance liability cash-flow streams in continuous time: Theory

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**Abstract**

We investigate fair (market-consistent and actuarial) valuation of insurance liability cash-flow streams in continuous time. We first consider one-period hedge-based valuations, where in the first step, an optimal dynamic hedge for the liability is set up, based on the assets traded in the market and a quadratic hedging objective, while in the second step, the remaining part of the claim is valuated via an actuarial valuation. Then, we extend this approach to a multi-period setting by backward iterations for a given discrete-time step $h$, and consider the continuous-time limit for $h \to 0$. We formally derive a partial differential equation for the valuation operator which satisfies the continuous-time limit of the multi-period, discrete-time iterations and prove that this valuation operator is actuarial and market-consistent. We show that our continuous-time fair valuation operator has a natural decomposition into the best estimate of the liability and a risk margin. The dynamic hedging strategy associated with the continuous-time fair valuation operator is also established. Finally, the valuation operator and the hedging strategy allow us to study the dynamics of the net asset value of the insurer.
**Keywords**: Optimal quadratic hedging, actuarial valuation, market-consistent valuation, fair valuation, partial differential equation, best estimate, risk margin, net asset value.

# 1 Introduction

During the past decades, major changes have taken place in the way insurance liabilities are valuated. This is due to the emergence of solvency regimes such as Swiss Solvency Test, Solvency II and C-ROSS (Chinese solvency regulation) which are built on the paradigm that valuations should be risk-based and take into account available information provided by financial markets. One of the key changes has been the requirement of determining *market-consistent* values for insurance liabilities in order to guarantee a better matching between assets and liabilities (see e.g. Albrecher et al. (2018)).

Standard *actuarial* valuation is typically based on a diversification argument which justifies applying the law of large numbers among independent policyholders who face i.i.d. risks. This valuation is performed under the real-world measure $\mathbb{P}$ and defined as the expectation plus an additional risk margin to cover any not fully diversified and non-diversifiable risk. Based on historical data, the actuarial valuation involves a subjective judgement concerning the choice of the model and its parameters. For a survey of the classical insurance approach, we refer to Kaas et al. (2008) for non-life and Norberg (2014) for life insurance.

Risk-neutral valuation is, on the other hand, market-driven, hence objective, and based on the idea of hedging and replication. By no-arbitrage arguments, prices of contingent claims can be expressed as expectations under a so-called *risk-neutral measure* $\mathbb{Q}$. This approach dates back to the seminal paper of Black & Scholes (1973) and was generalized to broad classes of processes. For an overview, see Delbaen & Schachermayer (2006).

A large branch of literature investigated valuations in a *market-consistent* setting, trying to extend the arbitrage-free pricing operators (initially defined in a complete market) to the general set of non-hedgeable claims. Several approaches were considered such as utility indifference pricing and $g$-expectations (Hodges & Neuberger (1989),Carmona (2009), Laeven & Stadje (2014)) or risk-minimization techniques (Föllmer & Schweizer (1988), Černỳ & Kallsen (2009), Delong (2013)). The notion of market-consistency has been recently formalized by several authors as an extension of the notion of cash-invariance to all hedgeable claims, see e.g. Malamud et al. (2008), Pelsser & Stadje (2014) and Dhaene et al. (2017).

It is important to notice that actuarial and risk-neutral valuations do not contradict each other, but are two types of valuations applied to different situations (diversifiable risks versus traded risks). In an insurance context, in which risks are partially traded and diversifiable, building a valuation framework which combines both approaches is primordial.

In this paper we investigate fair valuation of insurance liability cash-flow streams in continuous time, starting from the work of Dhaene et al. (2017), Barigou & Dhaene (2019) and Barigou et al. (2019) in discrete time. In line with these papers, we define a fair valuation as a valuation which is actuarial (mark-to-model for claims independent of the tradeable financial assets) and market-consistent (mark-to-market for any hedgeable part of a claim). We first consider the one-period hedge-based valuations introduced by Dhaene et al. (2017), where in the first step, an optimal dynamic hedge for the liability is set up, based on the assets traded in the market and a quadratic hedging objective, while in the second step, the remaining part of the claim is valuated via an actuarial valuation. Then, we extend this approach to a multi-period setting by backward iterations for a given discrete-time step $h$, and consider the continuous-time limit for $h \to 0$. We formally derive a partial differential equation (PDE) for the valuation operator which satisfies the continuous-time limit of the multi-period, discrete-time iterations and prove that this valuation operator is actuarial and market-consistent. We show that our continuous-time fair valuation operator has a natural decomposition into the best estimate of the liability and a risk margin. The dynamic hedging strategy associated with the continuous-time fair valuation operator is also established. Finally, the valuation operator and the hedging strategy allow us to study the dynamics of the net asset value of the insurer. We focus on the theory of fair valuation of insurance liability cash-flow streams in continuous time. Examples relevant for practice and more interpretations can be found in Delong et al. (2019).

We note that Pelsser (2010), Pelsser & Stadje (2014) and Pelsser & Ghalehjooghi (2016) also consider continuous-time valuation of insurance liabilities, taking into account both actuarial and market-consistent considerations. However, these papers investigate the valuation of a particular type of contingent claims at a fixed future time and do not explicitly study hedging strategies. We consider a general framework of insurance and financial cash-flows at random times up to a terminal time and derive an optimal, dynamic investment strategy for hedging the cash-flows. The one-period valuation operators proposed in this research are also different from the ones proposed in Pelsser (2010), Pelsser & Stadje (2014) and Pelsser & Ghalehjooghi (2016).

We should also mention the paper by Happ et al. (2015) who consider valuation of insurance liability cash-flow streams in a multi-period, discrete-time model. The best estimate of liability is obtained by sequential local risk minimization which describes the dynamically hedgeable part of the insurance cash-flows. These sequential risk minimization are obtained by orthogonal $L^2$ projections of the liability onto the space of hedgeable payoffs. Hence, the best estimate of the liability is the market value of the hedgeable pay-off derived from the orthogonal projection of the liability. The technical provision is then defined as the arbitrage-free price (including risk premiums for the non-hedgeable risks) of the insurance cash-flows. The multi-period valuation operator from Happ et al. (2015) has some similarities with our multi-period valuation operator, yet the constructions are different. Moreover, in this paper we focus on the continuous-time valuation operator and its properties.

This paper is organized as follows. Section 2 describes the financial and insurance model as well as the notions of market-consistent, actuarial and fair valuations and hedging strategies in continuous time. In Section 3, we present our one-period valuation problem and derive the optimal hedging strategy for the liability by quadratic hedging. Section 4 introduces the multi-period valuation operator by iterating the one-period valuation operator. Section 5 investigates the continuous-time limit of the multi-period valuation operator with quadratic one-period actuarial valuation operators (standard deviation and variance risk margins) used for valuating the non-hedgeable part of the liability. Section 6 discusses extensions beyond quadratic one-period actuarial valuations. All proofs are included in the appendix.

# 2 Financial and Insurance Model

Throughout the paper, we work on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ and a finite time horizon $T < \infty$. On the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ we define a standard two-dimensional Brownian motion $(W_1, W_2) = (W_1(t), W_2(t), 0 \leq t \leq T)$ and a càdlàg (right-continuous with left limits) counting process $N = (N(t), 0 \leq t \leq T)$. The Brownian motions $(W_1, W_2)$ are used to model the financial risk and the $\sigma$-algebra $\mathcal{F}_t^{W_1,W_2} = \sigma(W_1(u), W_2(u), u \in [0, t])$ contains all information on the evolution of the financial assets up to and including time $t$. The counting process $N$ is used to model the insurance risk and the $\sigma$-algebra $\mathcal{F}_t^N = \sigma(N(u), u \in [0, t])$ contains information on the number of in-force policies in the insurance portfolio up to and including time $t$. The insurance risk is not traded in the market, and the financial risk contains a tradeable and a non-tradeable component modelled with $W_1$ and $W_2$, respectively. We assume:

(A1) The subfiltrations $\mathbb{F}^{W_1,W_2} = (\mathcal{F}_t^{W_1,W_2})_{0\leq t\leq T}$ and $\mathbb{F}^N = (\mathcal{F}_t^N)_{0\leq t\leq T}$ are independent, and we set $\mathbb{F} = \mathbb{F}^{W_1,W_2} \times \mathbb{F}^N$.

Under assumption (A1) the financial risk is independent of the insurance risk.

## 2.1 The financial market

The financial market consists of a risk-free asset $R = (R(t), 0 \leq t \leq T)$ and two risky assets: $Y = (Y(t), 0 \leq t \leq T)$ and $F = (F(t), 0 \leq t \leq T)$. The price of the risk-free asset grows exponentially:

$$\frac{dR(t)}{R(t)} = rdt, \quad 0 \leq t \leq T, \quad R(0) = 1. \tag{2.1}$$

We assume that the prices of the risky assets $Y$ and $F$ are modelled with correlated geometric Brownian motions and follow the dynamics

$$\frac{dY(t)}{Y(t)} = \mu_Y dt + \sigma_Y dW_Y(t), \quad 0 \leq t \leq T, \quad Y(0) = y_0, \tag{2.2}$$

$$\frac{dF(t)}{F(t)} = \mu_F dt + \sigma_F dW_F(t), \quad 0 \leq t \leq T, \quad F(0) = f_0, \tag{2.3}$$

where $\mu_Y, \mu_F, \sigma_Y, \sigma_F$ are non-negative real numbers denoting the drifts and volatilities of the risky assets, respectively, while $(W_Y, W_F)$ denotes a correlated two-dimensional Brownian motion with correlation coefficient $\rho$. We define

$$W_Y(t) = W_1(t), \quad W_F(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t), \quad 0 \le t \le T. \tag{2.4}$$

where $(W_1, W_2)$ is the standard two-dimensional Brownian motion on $(\Omega, \mathbb{F}, \mathbb{P})$, i.e. $W_1$ is independent of $W_2$. We will use both $(W_Y, W_F)$ and $(W_1, W_2)$ in the sequel.

The insurance company can invest in the risk-free asset $R$ and in the risky asset $Y$. The risky asset $F$ is not available for trading. The insurance benefits are assumed to be linked to $F$. The risky asset $F$ can have different interpretations depending on the application, see Examples 1-2 below. Our intention is to consider correlated risky assets $Y$ and $F$ with $\rho \neq 0$. However, the case with $\rho = 0$ is also allowed, see Example 1.

## 2.2 The insurance model

The insurance company holds a homogeneous portfolio which consists of $n$ identical insurance policies. All policyholders have the same age (or are classified into the same age group) and are entitled to three types of benefits: a continuous annuity benefit $A$ paid in $[0, T]$ as long as the insured is alive, a death benefit $D$ paid at the moment that the insured dies (provided she or he dies in $[0, T]$) and a survival benefit $S$ paid at terminal time $T$ if the insured survives that time. The benefits $A, D$ and $S$ are allowed to be time-dependent and contingent on the values of the risky assets $(Y, F)$.

The process $N$ counts the number of deaths in the insurance portfolio. We assume:

(A2) The lifetimes of the policyholders $(\tau_k)_{k=1,\dots,n}$ at policy inception are independent and identically distributed with the survival function:

$$\mathbb{P}\big(\tau_k > t\big) = e^{-\int_0^t \lambda(s)ds}, \quad k = 1, \dots, n, \quad 0 \le t \le T.$$

The function $\lambda : [0, T] \mapsto (0, \infty)$ is continuously differentiable, i.e. $\lambda \in C^1([0, T])$, and strictly positive.

We have that

$$N(t) = \sum_{k=1}^{n} \mathbf{1}\{\tau_k \le t\}, \quad 0 \le t \le T.$$

The deterministic function $\lambda$ describes the mortality intensity of the policyholders in the insurance portfolio. We introduce the compensated counting process $\tilde{N} = (\tilde{N}(t), 0 \le t \le T)$:

$$\tilde{N}(t) = N(t) - \int_0^t (n - N(s-))\lambda(s)ds, \quad 0 \le t \le T,$$

which is an $\mathbb{F}$-martingale. By $N(t-)$ we mean $\lim_{s\to t^-} N(s)$. The compensated counting process will be used to construct the stochastic integral describing the non-hedgeable insurance risk. We also introduce the process $J = (J(t), 0 \leq t \leq T)$:

$$J(t) = n - N(t), \quad 0 \leq t \leq T,$$

which counts the number of in-force policies in the insurance portfolio.

The cash-flow stream of the benefit payments from the portfolio is modelled by the process $B = (B(t), 0 \leq t \leq T)$. This process is described by the following equation

$$\begin{aligned} B(t) &= \int_0^t J(u-)A(u, Y(u), F(u))du + \int_0^t D(u, Y(u), F(u))dN(u) \\ &\quad + J(T)S(Y(T), F(T))\mathbf{1}\{t = T\}, \quad 0 \leq t \leq T. \end{aligned} \tag{2.5}$$

The process $B$ is $\mathbb{F}$-adapted. The general benefit stream (2.5) was considered in Møller & Steffensen (2007) who studied market-consistent valuation methods in life insurance. By paying the benefits from the process $B$, we observe that the insurer is exposed to three sources of risk:

- Tradeable financial risk $Y$: The fluctuations of the risky asset $Y$ impact the payment process (2.5). This risk can be perfectly hedged (replicated) by dynamically trading in $(R, Y)$, i.e. we can construct a self-financing dynamic investment portfolio consisting of $(R, Y)$ such that its value is equal, in all scenarios, to the claims paid.

- Non-tradeable financial risk $F$: The variations of the risky asset $F$ impact the benefit stream (2.5) as well. This risk can be partially hedged by trading in $Y$, since $Y$ and $F$ are correlated. The higher the absolute value of the correlation parameter, the better the hedge. Consequently, the non-tradeable financial risk $F$ has two components: a hedgeable component and an independent, non-hedgeable component.

- Non-tradeable insurance risk $N$: The non-tradeable insurance risk arises since the policyholders die at random times and the death-related benefits have to be paid at unpredictable times. This risk cannot be hedged since it is assumed to be independent of the financial market.

**Example 1:** The insurer is only exposed to a terminal benefit $S(Y(T), F(T))$ which depends on the tradeable risky asset $Y(T)$ and the non-tradeable risky asset $F(T)$. The benefit stream (2.5) takes the form

$$B(t) = S(Y(T), F(T))\mathbf{1}\{t = T\}.$$

Such claims are common in financial mathematics as payoffs of European-type options on tradeable assets (e.g. exchange-traded stocks) or options on non-tradeable assets (e.g. electricity, temperature, etc.). In insurance, such payoffs are also considered by Pelsser (2010) as unit-linked contracts in which the non-tradeable asset $F$ represents an insurance process partially correlated to the stock $Y$.

In the examples above we would assume that $\rho \neq 0$. If $\rho = 0$, then the non-tradeable financial risk $F$ only includes an independent, non-hedgeable component, and $F$ could be interpreted as an independent non-financial risk (e.g. health status) modelled by a diffusion process, in contrast to the independent insurance risk $N$ which is modelled by a jump process. $\square$

**Example 2:** We consider a portfolio consisting of $n$ unit-linked insurance contracts in which each policyholder pays a single premium $P(0)$ at time 0. An initial fee $eP(0)$ is deducted from the premium to cover administrative costs, and the remaining amount $F(0) = P(0)(1-e)$ is invested by the insurer in the risky asset $Y$. The premium invested together with investment returns constitutes the so-called policyholder's fund. Since the insurer charges fees from the policyholder's fund, we define the following dynamics for the non-tradeable risky asset $F$:

$$\frac{dF(t)}{F(t)} = (\mu_Y - c)dt + \sigma_Y dW_Y(t),$$

where $c$ denotes the continuously deducted fee from the policyholder's fund to be paid for the guarantees stipulated in the unit-linked contract. For each policy, the insurer is exposed to death and survival put guarantees. In the benefit stream (2.5) we choose

$$D(t, Y(t), F(t)) = \max\{D^* - F(t), 0\}, \quad S(Y(T), F(T)) = \max\{S^* - F(T), 0\}.$$

Since the insurer collects the fees from the policyholders' funds continuously, in (2.5) we must set

$$A(t, Y(t), F(t)) = -cF(t).$$

$\square$

The benefit streams from Examples 1-2 and their fair prices are studied in Delong et al. (2019).

In this paper we assume:

(A3) The functions $A : [0,T]\times(0,\infty)\times(0,\infty) \mapsto [0,\infty)$, $D : [0,T]\times(0,\infty)\times(0,\infty) \mapsto [0,\infty)$ and $S : (0,\infty)\times(0,\infty) \mapsto [0,\infty)$ are Lipschitz continuous and have linear growth in

$(y, f)$:

$$\begin{aligned}
|A(t,y_1,f_1) - A(t,y_2,f_2)| &\leq K(|y_1-y_2| + |f_1-f_2|),\\
|A(t,y,f)| &\leq K(1+|y|+|f|),\\
|D(t,y_1,f_1) - D(t,y_2,f_2)| &\leq K(|y_1-y_2| + |f_1-f_2|),\\
|D(t,y,f)| &\leq K(1+|y|+|f|),\\
|S(y_1,f_1) - S(y_2,f_2)| &\leq K(|y_1-y_2| + |f_1-f_2|),\\
|S(y,f)| &\leq K(1+|y|+|f|).
\end{aligned}$$

Moreover, the functions $A$ and $D$ are continuous in $t$ and satisfy the conditions:

$$\begin{aligned}
|A(t_1,y,f) - A(t_2,y,f)| &\leq K|t_1-t_2|(1+|y|+|f|),\\
|D(t_1,y,f) - D(t_2,y,f)| &\leq K|t_1-t_2|(1+|y|+|f|).
\end{aligned}$$

We remark that the benefits $A, D, S$ paid at time $t$ depend on the current values of the assets $Y$ and $F$. In practice the benefits paid to policyholders very often depend on the returns earned by the backing assets over some time period. Since we want to use a PDE to characterize the fair value of the benefit stream $B$, we must choose a Markovian structure of the financial and insurance model.

## 2.3 Equivalent martingale measures

Let us introduce the class of equivalent martingale measures in our combined financial and insurance model. We define

$$\begin{aligned}
\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t &= \mathcal{M}^{\zeta,\chi}(t), \quad 0 \leq t \leq T,\\
\frac{d\mathcal{M}^{\zeta,\chi}(t)}{\mathcal{M}^{\zeta,\chi}(t-)} &= -\Big(\frac{\mu_Y - r}{\sigma_Y}\Big)dW_1(t) + \zeta(t)dW_2(t) + \chi(t)d\tilde{N}(t),
\end{aligned} \tag{2.6}$$

where $(\zeta, \chi)$ are predictable, Markov processes with respect to the natural filtration $\mathbb{F}^{Y,F,N} = \mathbb{F}^{Y,F} \times \mathbb{F}^{N}$ generated by $(Y, F, N)$, and they satisfy the conditions:

$$\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}}\Big[\int_t^T |\zeta(s)|^2 ds \big| \mathcal{F}_t^{Y,F,N}\Big] \leq K, \quad -1+\epsilon \leq |\chi(t)| \leq K, \quad 0 \leq t \leq T,\ \epsilon > 0.$$

By Lemma 1 in Morlais (2010) the process $\mathcal{M}^{\zeta,\chi}$ is a $\mathbb{P}$-martingale and can be used to define the equivalent probability measure $\mathbb{Q}$ such that the discounted process of the traded asset $(e^{-rt}Y(t), 0 \leq t \leq T)$ is a $\mathbb{Q}$-martingale. The process $\zeta$ is called the risk premium for the independent part of the non-tradeable financial risk $F$, and the process $\chi$ is called the risk premium for the independent non-tradeable insurance risk $N$.

Specifying the equivalent martingale measure $\mathbb{Q}$ for arbitrage-free pricing of the benefit stream $B$ is equivalent to specifying the risk premiums $(\zeta, \chi)$ for the non-tradeable financial and insurance risks. If we choose $\zeta(t) = \chi(t) = 0$ in (2.6), then we introduce

$$\begin{aligned} \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}|\mathcal{F}_t &= \hat{\mathcal{M}}(t), \quad 0 \le t \le T, \\ \frac{d\hat{\mathcal{M}}(t)}{\hat{\mathcal{M}}(t)} &= -\big(\frac{\mu_Y - r}{\sigma_Y}\big)dW_1(t). \end{aligned} \tag{2.7}$$

The measure $\hat{\mathbb{Q}}$ is a variance optimal martingale measure and a minimal martingale measure in our combined financial and insurance model, see Chapter 10.4.3 in Cont & Voltchkova (2005), Proposition 10.2.3 and Collorary 10.3.1 in Delong (2013). The measure $\hat{\mathbb{Q}}$ is also the unique equivalent martingale measure for the traded risky asset $Y$ in the complete market consisting only of $(R, Y)$. We will use $\hat{\mathbb{Q}}$ in the sequel.

## 2.4 Fair valuations and fair hedging strategies

We use the notions of *market-consistent, actuarial and fair* valuations and hedging strategies introduced by Dhaene et al. (2017) and further studied by Barigou & Dhaene (2019) and Barigou et al. (2019) in a multi-period, discrete time setting. Hereafter, we adapt these notions to our continuous-time model. To do so, we consider special cases of the benefit stream $B$. First, we consider the case in which the benefits are only contingent on the tradeable financial risk $Y$:

$$B^Y(t) = \int_0^t A(u, Y(u))du + S(Y(T))\mathbf{1}\{t = T\}, \quad 0 \le t \le T. \tag{2.8}$$

Secondly, we investigate the case in which the benefits are only contingent on the non-tradeable insurance risk $N$:

$$B^N(t) = \int_0^t J(u-)A(u)du + \int_0^t D(u)dN(u) + J(T)S\mathbf{1}\{t = T\}, \quad 0 \le t \le T. \tag{2.9}$$

Finally, we consider the benefits contingent only on the non-tradeable financial risk $F$ if the non-tradeable financial risk $F$ is independent of the tradeable financial risk $Y$:

$$B^{F,\perp\!\!\!\perp}(t) = \int_0^t A(u, F(u))du + S(F(T))\mathbf{1}\{t = T\}, \quad 0 \le t \le T, \quad \rho = 0. \tag{2.10}$$

If $\rho = 0$, then it is more appropriate to interpret $B^{F,\perp\!\!\!\perp}$ as an independent, non-tradeable, non-financial risk, see Example 1. The process $B^Y$ can be perfectly hedged by dynamically trading in $(R, Y)$ and will be called a *hedgeable* process. The processes $B^N$ and $B^{F,\perp\!\!\!\perp}$ are independent of the tradeable risky asset $Y$ and will be called *orthogonal* processes. A general orthogonal process will be denoted by $B^O$ and has the representation:

$$\begin{aligned} B^O(t) &= \int_0^t J(u-)A(u,F(u))du + \int_0^t D(u,F(u))dN(u) \\ &\quad + J(T)S(F(T))\mathbf{1}\{t=T\}, \quad 0 \le t \le T, \quad \rho = 0. \end{aligned} \tag{2.11}$$

Clearly, the process $B^Y$ is $\mathbb{F}^{W_1}$-adapted, the process $B^N$ is $\mathbb{F}^N$-adapted and the process $B^{F,\perp\!\!\!\perp}$ is $\mathbb{F}^{W_2}$-adapted.

**Definition 2.1.** *Let $\varphi_{B(t,T]}(t)$ denote the value at time $t \in [0,T]$ of the future claims from the process $B$. For each $t \in [0,T]$, the operator $\varphi_{B(t,T]}(t)$ maps a $\sigma\big(\mathbf{1}\{u \ge t\}W_1(u), \mathbf{1}\{u \ge t\}W_2(u), u \in [0,T]\big) \times \sigma\big(\mathbf{1}\{u \ge t\}N(u), u \in [0,T]\big)$-measurable random variable into an $\mathcal{F}_t$-measurable random variable. We will say that*

- *the valuation operator $\varphi$ is market-consistent if for any process $B$ and any hedgeable process $B^Y$, as defined in (2.8), we have that*

$$\begin{aligned} \varphi_{B(t,T]+B^Y(t,T]}(t) &= \varphi_{B(t,T]}(t) + \varphi_{B^Y(t,T]}(t), \\ &\textit{with} \\ \varphi_{B^Y(t,T]}(t) &= \mathbb{E}^{\hat{\mathbb{Q}}}\Big[\int_t^T e^{-r(u-t)}dB^Y(u)|\mathcal{F}_t^{W_1}\Big], \ 0 \le t \le T, \end{aligned} \tag{2.12}$$

  *where the equivalent martingale measure $\hat{\mathbb{Q}}$ is defined in* (2.7)*,*

- *the valuation operator $\varphi$ is actuarial if for any orthogonal process $B^O$, as defined in* (2.9)-(2.11)*, we have that*

$$\varphi_{B^O(t,T]}(t) = \mathbb{E}^{\mathbb{P}}\Big[\int_t^T e^{-r(u-t)}dB^O(u)|\mathcal{F}_t^{W_2} \times \mathcal{F}_t^N\Big] + RM^{act}_{B^O(t,T]}(t), \quad 0 \le t \le T, \tag{2.13}$$

  *where, for each $t \in [0,T]$, the operator $RM^{act}_{B^O(t,T]}(t)$ maps a $\sigma\big(\mathbf{1}\{u \ge t\}W_2(u), u \in [0,T]\big) \times \sigma\big(\mathbf{1}\{u \ge t\}N(u), u \in [0,T]\big)$-measurable random variable into an $\mathcal{F}_t^{W_2} \times \mathcal{F}_t^N$-measurable random variable. Moreover, the operator $RM^{act}_{B^N(t,T]}(t)$, restricted to $B^N$, maps a $\sigma\big(\mathbf{1}\{u \ge t\}N(u), u \in [0,T]\big)$-measurable random variable into an $\mathcal{F}_t^N$-measurable random variable, and the operator $RM^{act}_{B^{F,\perp\!\!\!\perp}(t,T]}(t)$, restricted to $B^{F,\perp\!\!\!\perp}$, maps a $\sigma\big(\mathbf{1}\{u \ge t\}W_2(u), u \in [0,T]\big)$-measurable random variable into an $\mathcal{F}_t^{W_2}$-measurable random variable,*

- *the valuation operator $\varphi$ is fair if it is market-consistent and actuarial.*

The definition of the valuation operator for the benefit stream as a measurable mapping is taken from Cheridito et al. (2006). The market-consistency condition (2.12) imposes that the claims which can be replicated by dynamically trading in $(R, Y)$ should be valuated at the price of their perfect hedge, since they do not carry any risk. The mark-to-model condition (2.13) corresponds to the traditional actuarial valuation of insurance liabilities. It postulates that any orthogonal process, which is independent of the tradeable financial assets, should be valuated by an operator which only takes into account the information about the orthogonal risk (with no reference to the financial market) and the value of an orthogonal process should be determined by the conditional expected value of the discounted future benefits and a risk loading.

Let us note that any valuation operator $\varphi$ can be decomposed into the conditional expected value, under the real-world measure $\mathbb{P}$, of the discounted future benefits and another valuation operator $RM$ which we call a *risk margin* valuation operator (the risk margin can be simply defined as the valuation operator minus the conditional expected value of the discounted benefits). Consequently, the risk margin valuation operator $RM$ is called actuarial if $RM$ reduces to an actuarial risk margin $RM^{act}$ as defined in (2.13) when applied to an orthogonal benefit stream $B^O$. Our Definition 2.1 of the risk margin $RM^{act}$ is very general. Let us consider $B^N$. We can have some obvious examples:

$$\begin{aligned} RM^{act}_{B^N(t,T]}(t) &= SD^{\mathbb{P}}\Big[\int_t^T e^{-r(u-t)}dB^N(u)|\mathcal{F}^N_t\Big], \\ RM^{act}_{B^N(t,T]}(t) &= VaR^{\mathbb{P}}_{1-\beta}\Big[\int_t^T e^{-r(u-t)}dB^N(u)|\mathcal{F}^N_t\Big], \end{aligned} \tag{2.14}$$

where $SD$ stands for standard deviation and $VaR$ stands for Value-at-Risk, as well as more sophisticated valuation operators:

$$RM^{act}_{B^N(t,T]}(t) = \mathbb{E}^{\mathbb{P}}\Big[\int_t^T e^{-r(u-t)}g(u,N(u))du|\mathcal{F}^N_t\Big], \tag{2.15}$$

where $g$ is a deterministic function used to quantify the *local* risk of the benefit stream $B^N$. Since the risk of $B^N$ is generated by the process $N$, the function $g$ only depends on $N$. The valuation operator (2.15) is an example of a so-called $g$-expectation and a dynamic risk measure, see Rosazza Gianin (2006).

After having considered desirable properties that a valuation operator should have, we introduce similar properties for the hedging operator.

**Definition 2.2.** *Let us consider a self-financing dynamic hedging portfolio consisting of the risky asset $Y$ and the risk-free asset $R$. Let $(\theta_{B(t,T]}(s), t \le s \le T)$ denote the hedging strategy in the risky asset $Y$ for the future claims from the process $B$ (the amount of money invested in the risky asset $Y$). We will say that*

- *the strategy $\theta$ is market-consistent if for any process $B$ and any hedgeable process $B^Y$*

*we have that*

$$\theta_{B(t,T]+B^Y(t,T]}(s) = \theta_{B(t,T]}(s) + v_y(s, Y(s))Y(s), \quad t \le s \le T, \tag{2.16}$$

*where* $v(t,y) = \mathbb{E}^{\hat{\mathbb{Q}}}[\int_t^T e^{-r(u-t)} dB^Y(u) | Y(t) = y]$*,* $v_y$ *denotes the derivative of* $v(t,y)$ *with respect to* $y$*, and the equivalent martingale measure* $\hat{\mathbb{Q}}$ *is defined in* (2.7)*,*

- *the strategy* $\theta$ *is actuarial if for any orthogonal process* $B^O$ *we have that*

$$\theta_{B^O(t,T]}(s) = 0, \quad t \le s \le T, \tag{2.17}$$

- *the strategy* $\theta$ *is fair if it is market-consistent and actuarial.*

The market-consistency condition (2.16) imposes the natural condition that any hedgeable claim should be hedged by its replicating hedging strategy. Let us remark that the second term in (2.16) is the delta-hedging strategy for $B^Y$, which is the replicating strategy for $B^Y$. The condition (2.17) imposes that any orthogonal process is hedged with a risk-free investment, which seems reasonable since the orthogonal process is independent of the tradeable financial asset $Y$. Hence, if the insurer is exposed to an orthogonal process, then the capital which protects the insurers against the orthogonal claims should be invested in the risk-free bank account[1].

We believe that a reasonable valuation operator and a hedging strategy should satisfy the properties which we postulate in Definitions 2.1-2.2. In the next sections, we construct such a valuation operator and a hedging strategy in a multi-period discrete-time model and in a continuous-time model.

# 3 The one-period valuation operator and dynamic hedging strategies

Let us start with the benefit stream $B$ which only includes claims at the terminal time $T$. In this case, the benefit stream (2.5) takes the form

$$B(t) = J(T)S(Y(T), F(T))\mathbf{1}\{t = T\}, \quad 0 \le t \le T.$$

We define

$$\triangle B(T) = J(T)S(Y(T), F(T)),$$

[1] We remark that if the interest rate were stochastic, then the capital which protects the insurer against the orthogonal claims would be invested in the (stochastic) risk-free bank account and in the available bonds in order to protect the insurer against the interest rate risk, since the price of the benefit stream and the risk margin are based on the discounted present value of the future claims

which specifies the benefits from the process $B$ to be paid at the terminal time $T$. We follow Dhaene et al. (2017) and we consider the one-period valuation operator $\varrho$ for the benefit stream $B$:

$$\varrho(B) = V_B(0) + \pi\Big((\triangle B(T) - V_B(T))e^{-rT}\Big), \tag{3.1}$$

where $V_B(t)$ denotes the time-$t$ value of a self-financing dynamic hedging portfolio for the claims from the process $B$ (for $t = 0, T$), and $\pi$ denotes a one-period actuarial valuation operator. The self-financing dynamic hedging portfolio consists of the risky asset $Y$ and the risk-free asset $R$ which are traded in the financial market. The idea of the valuation operator (3.1) is to split the valuation of the benefit stream $B$ into two parts: $V_B(0)$ gives the market value of the self-financing investment portfolio which hedges the tradeable financial risk of $B$, and $\pi$ gives the actuarial value of the remaining non-tradeable financial and insurance risks of $B$ which cannot be hedged. The valuation of $B$ is split into two parts but the valuation is not split from hedging since the initial value of the investment portfolio, which hedges the hedgeable part of $B$, constitutes a part of the price of $B$. The remaining part of the price of $B$ is related to the non-hedgeable part of $B$ which is left after the application of the hedging portfolio, and only this part of the valuation operator can be set independently from hedging. Two decisions need to be made when choosing the valuation operator (3.1): first, the method to define the hedging portfolio; second, the choice of the one-period actuarial valuation.

Let us discuss the hedging portfolio and the hedging problem for our benefit stream (2.5). Dhaene et al. (2017) suggest to minimize the mean-square hedging error under the real-world measure at the terminal time to derive the static hedging portfolio in their one-period model. In continuous-time models, there are three, fundamentally different, approaches to minimize the hedging error in a mean-square sense: quadratic hedging under an equivalent martingale measure, quadratic hedging under the real-world measure and local risk-minimization, see e.g. Chapter 10.4 in Cont & Voltchkova (2005) and Chapter 10 in Delong (2013). We only consider quadratic hedging under an equivalent martingale measure and under the real-world measure since locally risk minimizing strategies are not self-financing.

We separate the benefit stream $B$, which consists of benefits to be paid in $(0, T]$, into two parts. Let $B(0, T)$ denote the benefits from $B$ to be paid in $(0, T]$ excluding the terminal benefits to be paid at time $T$, and $\triangle B(T)$ denote the terminal benefits from $B$ to be paid at time $T$. We clearly have

$$\begin{aligned} B(0,T) &= \int_0^T J(t-)A(t, Y(t), F(t))dt + \int_0^T D(t, Y(t), F(t))dN(t), \\ \triangle B(T) &= J(T)S(Y(T), F(T)), \\ B(T) &= B(0,T] = B(0,T) + \triangle B(T). \end{aligned} \tag{3.2}$$

Let $\theta = (\theta(t), 0 \le t \le T)$ denote an $\mathbb{F}$-predictable stochastic process which describes the amount invested in the risky asset $Y$. Let $V^\theta = (V^\theta(t), 0 \le t \le T)$ denote the value of the self-financing dynamic hedging portfolio under the strategy $\theta$. The process $V^\theta$ can also be interpreted as the wealth process of the insurer. The amount invested in the risk-free asset $R$ at time $t$ is given by $V^\theta(t) - \theta(t)$. The dynamics of the self-financing dynamic hedging portfolio $V^\theta$ is described with the SDE:

$$\begin{aligned} dV^\theta(t) &= \theta(t)(\mu_Y dt + \sigma_Y dW_Y(t)) + (V^\theta(t) - \theta(t))r dt \\ &\quad - J(t-)A(t, Y(t), F(t))dt - D(t, Y(t), F(t))dN(t), \end{aligned} \tag{3.3}$$

and the terminal claims $\triangle B(T)$ are subtracted from $V^\theta(T)$ at time $T$.

Let us start with quadratic hedging under an equivalent martingale measure. Let us choose $\mathbb{Q}$ from the set of equivalent martingale measures defined in (2.6). We can find the self-financing hedging portfolio and the hedging strategy which minimize the mean-square hedging error at the terminal time under the equivalent martingale measure $\mathbb{Q}$, i.e. we solve:

$$\begin{aligned} &\inf_\theta \mathbb{E}^{\mathbb{Q}}\big[|\triangle B(T) - V^\theta(T)|^2\big], \quad \mathbb{Q} \sim \mathbb{P}, \\ &V^\theta \text{ satisfies the dynamics (3.3).} \end{aligned} \tag{3.4}$$

The optimal hedging strategy and the optimal hedging portfolio are denoted by $\theta^*_{B(0,T),\triangle B(T)}$ and $V^*_{B(0,T),\triangle B(T)}$, or simply by $\theta^*_B$ and $V^*_B$. We will mostly use $B(0,T)$ and $\triangle B(T)$ instead of $B$ (or $B(0,T]$), since we need to separate the claims as in (3.2).

**Proposition 3.1.** *We consider the claims from the process $B$, which are separated in accordance with* (3.2)*. Suppose that (A1)-(A3) hold and let us choose an equivalent martingale measure $\mathbb{Q}$ from the set defined in* (2.6)*. We assume there exist functions $(v^k)_{k=0,...,n}$*

$$\begin{aligned} v^k(t,y,f) &= \mathbb{E}^{\mathbb{Q}}_{t,y,f,k}\Big[\int_t^T e^{-r(u-t)} dB(u)\Big], \\ &\qquad (t,y,f) \in [0,T] \times (0,\infty) \times (0,\infty),\ k \in \{0,...,n\}, \end{aligned}$$

*such that $v^k \in \mathcal{C}^{1,2,2}([0,T) \times (0,\infty) \times (0,\infty)) \cup \mathcal{C}([0,T] \times (0,\infty) \times (0,\infty))$, for each $k \in \{0,...,n\}$.*

*We consider the optimization problem* (3.4) *under the equivalent martingale measure $\mathbb{Q}$. The initial value of the self-financing hedging portfolio is given by*

$$V^*_{B(0,T),\triangle B(T)}(0) = v^n(0, Y(0), F(0)), \tag{3.5}$$

*and the optimal dynamic hedging strategy is given by*

$$\begin{aligned} \theta^*_{B(0,T),\triangle B(T)}(t) &= v_y^{J(t-)}(t, Y(t), F(t))Y(t) \\ &\quad + v_f^{J(t-)}(t, Y(t), F(t))F(t)\frac{\sigma_F}{\sigma_Y}\rho, \quad 0 \le t \le T. \end{aligned} \tag{3.6}$$

**Remark 3.1.** *a) The expected value* $\mathbb{E}_{t,y,f,k}[\cdot]$ *denotes the conditional expected value* $\mathbb{E}[\cdot|Y(t) = y, F(t) = f, J(t) = k]$.
*b) The collection of functions* $(v^k)_{k=0,...,n}$ *gives us the arbitrage-free price of the benefit stream* $B$ *under the equivalent martingale measure* $\mathbb{Q}$. *It is standard in financial mathematics to assume that such functions are smooth, or that the solutions to the PDEs used for option pricing are smooth, and satisfy some growth conditions, see e.g. Heath & Schweizer (2000), Proposition 4.3 in El Karoui et al. (1997) or Proposition 1 in Cont & Voltchkova (2005).*
*c) If we choose* $\mathbb{Q} = \hat{\mathbb{Q}}$ *defined in* (2.7), *then the optimal amount of wealth invested in the risky asset* $Y$ *has the same form as the locally risk minimizing strategy for the risky asset* $Y$, *see Chapter 10.3 in Delong (2013). However, as already pointed out, the locally risk minimizing strategy and the corresponding hedging portfolio are not self-financing since in this framework we would have* $V^*_{B(0,T),\triangle B(T)}(t) = v^{J(t)}(Y(t), F(t))$, *for all* $t \in [0, T]$, *whereas for our optimal self-financing hedging portfolio the equality* (3.5) *only holds at time* $t = 0$.
*d) In the case of orthogonal processes* (2.9)-(2.11), *it is natural to restrict the set of equivalent martingale measures* (2.6) *used in the hedging objective* (3.4), *and we should assume that*

*(A4) We choose* $\mathbb{F}^F \times \mathbb{F}^N$*-predictable* $\zeta$ *and* $\chi$ *if we hedge* $B^O$. *Moreover, we choose an* $\mathbb{F}^F$*-predictable* $\zeta$ *if we hedge* $B^{F,\perp}$, *and we choose an* $\mathbb{F}^N$*-predictable* $\chi$ *if we hedge* $B^N$.

Let us recall that the proofs can be found in the appendix.

The quadratic hedging objective (3.4) under an equivalent martingale measure might be criticized since the insurer is interested in valuating losses and profits in the real-world. One can construct the self-financing hedging portfolio and the hedging strategy by minimizing the mean-square hedging error under the real-world measure:

$$\begin{aligned} &\inf_{\theta} \mathbb{E}^{\mathbb{P}}\big[|\triangle B(T) - V^{\theta}(T)|^2\big], \\ &V^{\theta} \text{ satisfies the dynamics (3.3).} \end{aligned} \tag{3.7}$$

**Proposition 3.2.** *We consider the claims from the process* $B$, *which are separated in accordance with* (3.2). *Suppose that (A1)-(A3) hold and let* $\hat{\mathbb{Q}}$ *denote the equivalent martingale measure defined in* (2.7). *We assume there exist functions* $(v^k)_{k=0,...,n}$

$$\begin{aligned} v^k(t, y, f) \;\; &= \;\; \mathbb{E}^{\hat{\mathbb{Q}}}_{t,y,f,k}\Big[\int_t^T e^{-r(u-t)} dB(u)\Big], \\ & \qquad (t, y, f) \in [0, T] \times (0, \infty) \times (0, \infty), \;\; k \in \{0, ..., n\}, \end{aligned}$$

*such that* $v^k \in \mathcal{C}^{1,2,2}([0, T) \times (0, \infty) \times (0, \infty)) \cup \mathcal{C}([0, T] \times (0, \infty) \times (0, \infty))$, *for each* $k \in \{0, ..., n\}$.
*We consider the optimization problem* (3.7) *under the real-world measure. The initial value of the self-financing hedging portfolio is given by*

$$V^*_{B(0,T),\triangle B(T)}(0) = v^n(0, Y(0), F(0)), \tag{3.8}$$

*and the optimal dynamic hedging strategy is given by*

$$\theta^*_{B(0,T),\triangle B(T)}(t) = v_y^{J(t-)}(t, Y(t), F(t))Y(t) + v_f^{J(t-)}(t, Y(t), F(t))F(t)\frac{\sigma_F}{\sigma_Y}\rho$$
$$-\Big(\frac{\mu_Y - r}{\sigma_Y^2}\Big)\Big(V^*_{B(0,T),\triangle B(T)}(t-) - v^{J(t-)}(t, Y(t), F(t))\Big) \quad 0 \le t \le T. \tag{3.9}$$

Let us compare the optimal hedging strategies (3.9) and (3.6).

**Proposition 3.3.** *Let (A1)-(A4) hold.*
*(i) The optimal dynamic hedging strategy from Proposition 3.1 derived by minimizing the mean-square hedging error under a martingale measure is market-consistent and actuarial,*
*(ii) The optimal dynamic hedging strategy from Proposition 3.2 derived by minimizing the mean-square hedging error under the real-world measure is market-consistent but not actuarial.*

The propositions above justify our choice of the hedging error under an equivalent martingale measure as the objective used in constructing the dynamic hedging strategy for the benefit stream $B$, at least in the class of quadratic hedging errors and self-financing hedging portfolios (see the example in the proof of Proposition 3.3 in the Appendix to gain an intuition why the quadratic hedging under the real-world measure leads to a strategy which is not actuarial). To support our choice we also note that the optimal dynamic hedging strategy from Proposition 3.1 is a delta-hedging strategy - a type of hedging strategies commonly used in practice.

In the sequel we assume that the hedging strategy and the self-financing hedging portfolio in the valuation operator (3.1) are derived by solving the optimization problem (3.4). Moreover, we decide to choose the equivalent martingale measure $\hat{\mathbb{Q}}$, defined in (2.7), when we solve the optimal hedging problem (3.4). Let us recall that the idea of the valuation operator (3.1) is to valuate the hedgeable and the non-hedgeable part of $B$ separately. From a practical point of view we strongly believe that it is reasonable not to include any assumptions on the risk premiums for the non-hedgeable risks when we solve our hedging problems and set the value of the hedging portfolio as the price of the hedgeable part of the benefit stream (i.e. we should choose $\zeta(t) = \chi(t) = 0$ in (2.6)). We recall that the choice of the risk premiums in (2.6) for pricing the non-hedgeable risks is subjective and we believe that these risk premiums should be implied by the subjective one-period actuarial valuation operator rather than being included in the equivalent martingale measure $\mathbb{Q}$ used for determining the hedging portfolio. This way we can disentangle hedgeable and non-hedgeable parts of the benefit stream and price them separately.

**Proposition 3.4.** *We consider the one-period valuation operator* (3.1). *Let (A1)-(A3) hold. The self-financing dynamic hedging portfolio and the dynamic hedging strategy are characterized in Proposition 3.1 where we choose the equivalent martingale measure $\hat{\mathbb{Q}}$ defined in*

(2.7). *We assume that the one-period actuarial valuation operator* $\pi$ *satisfies the conditions of normalization and translation-invariance:*

$$\pi(0) = 0, \quad \pi(\xi + a) = \pi(\xi) + a, \tag{3.10}$$

*for any random variable* $\xi$ *and constant* $a$. *The one-period valuation operator* (3.1) *is market-consistent and actuarial, hence it is fair.*

We find it more convenient to represent the actuarial valuation operator $\pi$ as the expected value operator under the real-world measure $\mathbb{P}$ and the actuarial risk margin, i.e. we assume

$$\pi(\xi) = \mathbb{E}^{\mathbb{P}}[\xi] + RM[\xi], \tag{3.11}$$

where the risk margin valuation operator $RM$ is an actuarial risk margin valuation operator which satisfies (2.13). The actuarial risk margin $RM$ fulfills the conditions implied by (3.10), i.e.

$$RM[0] = 0, \quad RM[\xi + a] = RM[\xi].$$

We will focus on standard deviation and variance under $\mathbb{P}$ as the one-period actuarial risk margins.

# 4 The multi-period valuation operator

In this section we introduce the multi-period valuation operator by iterating the one-period valuation operator $\varrho$ from Proposition 3.4. We follow the idea from Barigou & Dhaene (2019) and Barigou et al. (2019). Let $\mathcal{T} = \{0, h, ..., T-h, T\}$ with fixed $h$. The price $\varphi_{B(t,T]}(t)$ of the future benefit payments from the process $B$ at time $t \in \mathcal{T}$ is defined by the following backward iterations:

$$\begin{aligned}
\varphi_{\triangle B(T)}(T) &= \triangle B(T), \\
\varphi_{B(t,T]}(t) &= \varrho_t\Big(\int_t^{t+h} d\tilde{B}^{t,h}(s)\Big), \quad t = 0, h, ..., T-h, \\
\tilde{B}^{t,h}(s) &= \int_t^s J(u-)A(u, Y(u), F(u))du + \int_t^s D(u, Y(u), F(u))dN(u) \\
&\quad + \varphi_{B(t+h,T]}(t+h)\mathbf{1}\{s = t+h\}, \quad t \le s \le t+h,
\end{aligned} \tag{4.1}$$

where the valuation operator $\varrho_t$ is defined in Proposition 3.4, and it is now defined conditional on the information available in the $\sigma$-algebra $\mathcal{F}_t$. In the sequel $\varphi_{B(t,T]}(t)$ is simply denoted by $\varphi(t)$.

In order to apply the iterations (4.1) and calculate the price $\varphi(t)$ at $t \in \mathcal{T}$, we have to solve a sequence of optimal hedging problems of the form (3.4) in our multi-period model.

We remark that the insurer, in the period $[t, t+h]$, has to optimally hedge the claims from the process $B$ which arrive on $(t, t+h]$ and the value of the future claims $\varphi(t+h)$ at the terminal time $t+h$. The solutions to these hedging problems are given by Proposition 3.1 if we choose $\hat{\mathbb{Q}}$.

By the Markov property we have that $\varphi(t) = \varphi^{J(t)}(t, Y(t), F(t))$. Let

$$
\begin{aligned}
v^k(s,y,f) &= \mathbb{E}^{\hat{\mathbb{Q}}}_{s,y,f,k}\Big[\int_s^{t+h} e^{-r(u-s)}dB(u) + e^{-r(t+h-s)}\varphi^{J(t+h)}(t+h, Y(t+h), F(t+h))\Big], \\
&\qquad (s,y,f) \in [t,t+h] \times (0,\infty) \times (0,\infty),\ k \in \{0,...,n\}. \qquad (4.2)
\end{aligned}
$$

As in (3.2) we separate the benefit stream $\tilde{B}^{t,h}$ into $\tilde{B}(t,t+h)$ and $\triangle\tilde{B}(t+h)$. We have

$$
\begin{aligned}
\tilde{B}(t,t+h) &= \int_t^{t+h} J(u-)A(u,Y(u),F(u))du \\
&\quad + \int_t^{t+h} D(u,Y(u),F(u))dN(u) = B(t,t+h), \\
\triangle\tilde{B}(t+h) &= \varphi_{B(t+h,T]}(t+h).
\end{aligned}
$$

By Proposition 3.1, the optimal hedging strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$ for the period $[t,t+h]$ is given by

$$
\begin{aligned}
\theta^*_{B(t,t+h),\varphi(t+h)}(s) &= v_y^{J(s-)}(s,Y(s),F(s))Y(s) \\
&\quad + v_f^{J(s-)}(s,Y(s),F(s))F(s)\frac{\sigma_F}{\sigma_Y}\rho, \quad t \le s \le t+h. \qquad (4.3)
\end{aligned}
$$

By (4.1) and Proposition 3.4, the multi-period valuation operator is defined by the following iterations:

$$
\begin{aligned}
\varphi(t) &= V^*_{B(t,t+h),\varphi(t+h)}(t) \\
&\quad + \pi_t\Big(\big(\varphi(t+h) - V^*_{B(t,t+h),\varphi(t+h)}(t+h)\big)e^{-rh}\Big), \quad t = 0, h, ..., T-h,
\end{aligned}
$$

where $V^*_{B(t,t+h),\varphi(t+h)}$ denotes the optimal self-financing hedging portfolio given by (3.3) on $[t, t+h]$ with the optimal hedging strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$. Using the decomposition of the one-period actuarial valuation operator (3.11), we end up with the backward, discrete-time pricing equation:

$$
\begin{aligned}
\varphi(t) = V^*_{B(t,t+h),\varphi(t+h)}(t) + \mathbb{E}^{\mathbb{P}}\Big[\Big(\varphi(t+h) - V^*_{B(t,t+h),\varphi(t+h)}(t+h)\Big)e^{-rh}|\mathcal{F}_t\Big] \\
+RM\Big[\Big(\varphi(t+h) - V^*_{B(t,t+h),\varphi(t+h)}(t+h)\Big)e^{-rh}|\mathcal{F}_t\Big], \quad t = 0, h, ..., T-h. \qquad (4.4)
\end{aligned}
$$

The first two terms in (4.4) can be interpreted as the best estimate of the future claims generated, respectively, by the hedgeable and non-hedgeable risks of the benefit stream

$B$ during the period $(t, t+h]$. The initial value of the self-financing hedging portfolio $V^{*}_{B(t,t+h),\varphi(t+h)}(t)$ gives us the market cost (under zero risk premiums for the non-tradeable risks) of the investment strategy which replicates the hedgeable part of the benefit stream $B$ on $(t, t+h]$ and the claim $\varphi(t+h)$ at $t+h$. The expected value operator provides the expected cost, under the real-world measure $\mathbb{P}$, of the non-hedgeable claims which remain after the application of the hedging portfolio $V^{*}_{B(t,t+h),\varphi(t+h)}(t)$. The third term in (4.4), the risk margin, describes an additional capital which should be used by the insurer to cover the non-hedgeable part of the liability on $(t, t+h]$ in adverse scenarios when the capital determined by the first two terms is not sufficient to pay the claims from $\int_t^{t+h} dB$ and $\varphi(t+h)$. In the next section we will show that this desired decomposition of the value of the benefit stream also holds for our continuous-time valuation operator.

Let us introduce the discrete-time process

$$X^{[t,t+h]}(s) = \varphi(s) - V^{*}_{B(t,t+h),\varphi(t+h)}(s), \quad s \in \{t, t+h\}. \tag{4.5}$$

The process $\varphi$ should give us a *fair* price of the future benefits, or in other words it should give us the reserve the insurer should hold in order to cover the future benefits. Consequently, the process $NAV(s) = -X(s)$ determines the excess of the assets over the technical provision, and is called the net asset value. Using the assumptions on the actuarial risk margin from Proposition 3.4 and equation (4.5), we can transform our backward, discrete-time pricing equation (4.4) into a slightly different form:

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\big[X^{[t,t+h]}(t+h)e^{-rh} - X^{[t,t+h]}(t)|\mathcal{F}_t\big] \\ &\quad + RM\big[X^{[t,t+h]}(t+h)e^{-rh} - X^{[t,t+h]}(t)|\mathcal{F}_t\big] = 0, \quad t = 0, h, ..., T-h, \end{aligned} \tag{4.6}$$

or

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[NAV(t+h)e^{-rh} - NAV(t)|\mathcal{F}_t\right] \\ &= RM\left[NAV(t) - NAV(t+h)e^{-rh}|\mathcal{F}_t\right], \quad t = 0, h, ..., T-h. \end{aligned} \tag{4.7}$$

We observe that the insurance company grows at the expected rate higher than the risk-free rate $r$, and the expected growth rate is related to the value of the risk margin which covers the non-hedgeable risks, since

$$\begin{aligned} &RM\left[NAV(t) - NAV(t+h)e^{-rh}|\mathcal{F}_t\right] \\ &= RM\left[\big(\varphi(t+h) - V^{*}_{B(t,t+h),\varphi(t+h)}(t+h)\big)e^{-rh}|\mathcal{F}_t\right] > 0. \end{aligned} \tag{4.8}$$

The insurer at time $t+h$ should earn the risk margin which is included in the technical provision at time $t$, and covers the non-hedgeable risks in $(t, t+h]$, since the risk margin represents a safety buffer over the best estimate of the liability.

In practical applications our multi-period valuation operator would work as follows. At time $t$ the insurer sets the hedging portfolio $V^*_{B(t,t+h),\varphi(t+h)}(t)$ on the asset side. The valuation operator (4.4) gives us the price of the future benefits or, in other words, the technical provision which the insurer should set at time $t$ on the liability side. Since $\varphi_{B(t,T]}(t) > V^*_{B(t,t+h),\varphi(t+h)}(t)$ due to the actuarial risk margin applied to the non-hedgeable risks (and possible differences between the expected value of the future benefits and the expected future value of the hedging portfolio), the insurer must set an additional capital on the asset side equal to $\varphi_{B(t,T]}(t) - V^*_{B(t,t+h),\varphi(t+h)}(t)$ at time $t$. This additional capital is invested in the risk-free bank account. The net asset value at time $t$ is equal to zero. During the period $(t, t+h]$, the self-financing hedging portfolio $V^*_{B(t,t+h),\varphi(t+h)}$ is dynamically rebalanced in accordance with the strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$, the capital $\varphi_{B(t,T]}(t) - V^*_{B(t,t+h),\varphi(t+h)}(t)$ earns the risk-free rate and the insurer covers the benefits from the process $\int_t^{t+h} dB(s)$. We remark that the risk-free investment of $\varphi_{B(t,T]}(t) - V^*_{B(t,t+h),\varphi(t+h)}(t)$ means that the risk margin which we need to keep over the period $(t, t+h]$ is invested in the risk-free bank-account. When applying the hedging strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$, we try to hedge the benefits to be paid in $(t, t+h]$ and the price $\varphi(t+h)$ which includes all future risk margins which we need to keep after time $t+h$. This means that if the insurer is exposed to the non-hedgeable financial risk $F$ correlated with the risky asset $Y$ and keeps a risk margin against this risk, then a part of the total risk margin included in the price of the benefit stream $\varphi_{B(t,T]}$ at time $t$ is invested in the risky asset $Y$. This will be better seen in the next section. At time $t+h$ the price of the future benefits $\varphi_{B(t+h,T]}(t+h)$ and the net asset value are recalculated. The capital from the risk margin, set at time $t$ for the period $(t, t+h]$, is used only in adverse scenarios to cover the non-hedgeable benefits from $\int_t^{t+h} dB(s)$ and the non-hedgeable change in the price of the future benefits: $\varphi_{B(t+h,T]}(t+h) - \varphi_{B(t,T]}(t)$. In the average scenario the insurer earns the risk margin, accumulated with the risk-free rate, in the period $[t, t+h]$. This description confirms what we observe in practice. Similar conclusions will be derived in the next section in the continuous-time model. We remark that adding the additional capital $\varphi_{B(t,T]}(t) - V^*_{B(t,t+h),\varphi(t+h)}(t)$ to the assets and investing it in the risk-free bank account does not change $NAV(t+h)e^{-rh} - NAV(t)$, hence (4.6)-(4.8) remain valid.

# 5 The continuous-time limit of the multi-period valuation operator

We would like to extend the definition of the price $\varphi_{B(t,T]}(t)$ from times $t \in \{0, h, ..., T-h, T\}$ to all times $t \in [0, T]$. The goal is to take $h \to 0$ in (4.4) and transform the backward, discrete-time equation (4.4) into a continuous-time equation (differential equation) with a terminal condition.

We fix the period $[t, t+h]$. We extend (4.5) and we introduce the continuous-time process

$$X^{[t,t+h]}(s) = \varphi(s) - V^*_{B(t,t+h),\varphi(t+h)}(s), \quad t \leq s \leq t+h, \tag{5.1}$$

where $\varphi$ denotes the continuous-time valuation operator which gives the price of the future benefit payments from the process $B$ at any time $t \in [0,T]$. We characterize the continuous-time valuation operator $\varphi$ as a function which satisfies the continuous-time limit of the discrete-time pricing equation (4.6) as $h \to 0$.

By the Markov property we have that $\varphi(t) = \varphi^{J(t)}(t, Y(t), F(t))$. We can write the dynamics for the process $X^{[t,t+h]}$ on $[t, t+h]$. Applying Itô's lemma, we find that

$$\begin{aligned}
dX^{[t,t+h]}(s) = & \Big\{ (\varphi_t^{J(s-)}(s, Y(s), F(s)) + \varphi_y^{J(s-)}(s, Y(s), F(s)) Y(s) \mu_Y \\
& + \varphi_f^{J(s-)}(s, Y(s), F(s)) F(s) \mu_F + \varphi_{yf}^{J(s-)}(s, Y(s), F(s)) Y(s) F(s) \sigma_Y \sigma_F \rho \\
& + \frac{1}{2} \varphi_{yy}^{J(s-)}(s, Y(s), F(s)) Y^2(s) \sigma_Y^2 + \frac{1}{2} \varphi_{ff}^{J(s-)}(s, Y(s), F(s)) F^2(s) \sigma_F^2 \\
& - \theta^*_{B(t,t+h),\varphi(t+h)}(s)(\mu_Y - r) - V^*_{B(t,t+h),\varphi(t+h)}(s) r + J(s-) A(s, Y(s), F(s)) \Big\} ds \\
& + \Big( \varphi_y^{J(s-)}(s, Y(s), F(s)) Y(s) - \theta^*_{B(t,t+h),\varphi(t+h)}(s) \Big) \sigma_Y dW_Y(s) \\
& + \varphi_f^{J(s-)}(s, Y(s), F(s)) F(s) \sigma_F dW_F(s) + D(s, Y(s), F(s)) dN(s) \\
& + \Big( \varphi^{J(s-)-1}(s, Y(s), F(s)) - \varphi^{J(s-)}(s, Y(s), F(s)) \Big) dN(s) \\
= & \Big\{ X^{[t,t+h]}(s) r + \varphi_t^{J(s-)}(s, Y(s), F(s)) + \varphi_y^{J(s-)}(s, Y(s), F(s)) Y(s) \mu_Y \\
& + \varphi_f^{J(s-)}(s, Y(s), F(s)) F(s) \mu_F + \varphi_{yf}^{J(s-)}(s, Y(s), F(s)) Y(s) F(s) \sigma_Y \sigma_F \rho \\
& + \frac{1}{2} \varphi_{yy}^{J(s-)}(s, Y(s), F(s)) Y^2(s) \sigma_Y^2 + \frac{1}{2} \varphi_{ff}^{J(s-)}(s, Y(s), F(s)) F^2(s) \sigma_F^2 \\
& - \theta^*_{B(t,t+h),\varphi(t+h)}(s)(\mu_Y - r) - \varphi^{J(s-)}(s, Y(s), F(s)) r + J(s-) A(s, Y(s), F(s)) \\
& + \Big( \varphi^{J(s-)-1}(s, Y(s), F(s)) + D(s, Y(s), F(s)) - \varphi^{J(s-)}(s, Y(s), F(s)) \Big) J(s-) \lambda(s) \Big\} ds \\
& + \Big( \varphi_y^{J(s-)}(s, Y(s), F(s)) Y(s) + \varphi_f^{J(s-)}(s, Y(s), F(s)) F(s) \frac{\sigma_F}{\sigma_Y} \rho \\
& \quad - \theta^*_{B(t,t+h),\varphi(t+h)}(s) \Big) \sigma_Y dW_1(s) \\
& + \varphi_f^{J(s-)}(s, Y(s), F(s)) F(s) \sigma_F \sqrt{1-\rho^2} dW_2(s) \\
& + \Big( \varphi^{J(s-)-1}(s, Y(s), F(s)) + D(s, Y(s), F(s)) - \varphi^{J(s-)}(s, Y(s), F(s)) \Big) d\tilde{N}(s), \\
& \qquad t \leq s \leq t+h.
\end{aligned} \tag{5.2}$$

Looking at (5.2), we can conclude that the risk of the process $X^{[t,t+h]}$ is induced by the three stochastic integrals with respect to the Brownian motions and the compensated counting process. From (4.3) we expect that

$$\begin{aligned}\theta^*_{B(t,t+h),\varphi(t+h)}(s) \;\sim\;& \varphi_y^{J(s-)}(s,Y(s),F(s))Y(s)\\ &+\varphi_f^{J(s-)}(s,Y(s),F(s))F(s)\frac{\sigma_F}{\sigma_Y}\rho, \quad t\le s\le t+h,\end{aligned}$$

for sufficiently small $h$. Consequently, for sufficiently small $h$, the risk of the process $X^{[t,t+h]}$ is only induced by the two stochastic integrals with respect to the Brownian motion $W_2$ and the compensated counting process $\tilde{N}$. These two stochastic integrals cannot be hedged by trading in $Y$ in the financial market – they model the non-hedgeable financial and insurance risks to which the insurer is exposed. Consequently, the actuarial risk margin valuation operator in the limit $h\to 0$ in (4.6) should only be applied to the second and third stochastic integral from (5.2).

It is common to measure the risk of a stochastic process with its quadratic variation, at least the risk of a process in a quadratic sense. Hence, we expect that the actuarial risk margin valuation operator in the limit $h\to 0$ in (4.6) should act on the integrals:

$$\begin{aligned}&\Big[\int_t^{\cdot}\varphi_f^{J(u-)}(u,Y(u),F(u))F(u)\sigma_F\sqrt{1-\rho^2}dW_2(u)du\Big]\\ &=\int_t^{\cdot}\big(\varphi_f^{J(u-)}(u,Y(u),F(u))\big)^2F^2(u)\sigma_F^2\left(1-\rho^2\right)du,\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}&\Big[\int_t^{\cdot}\big(\varphi^{J(u-)-1}(u,Y(u),F(u))+D(u,Y(u),F(u))-\varphi^{J(u-)}(u,Y(u),F(u))\big)d\tilde{N}(u)\Big]\\ &=\int_t^{\cdot}\big(\varphi^{J(u-)-1}(u,Y(u),F(u))+D(u,Y(u),F(u))-\varphi^{J(u-)}(u,Y(u),F(u))\big)^2dN(u)\end{aligned}\tag{5.4}$$

The first integral (5.3) measures (in a quadratic sense) the non-hedgeable risk that the value of the benefit payments changes due to a change in the independent component of the risky asset $F$. The integrand in (5.3) is the delta-hedging perfect replication strategy for the independent component of the risky asset $F$. This delta-hedging strategy cannot be applied by the insurer since $F$ is not traded. The second integral (5.4) measures (in a quadratic sense) the non-hedgeable risk that a policyholder dies: in case of death, the death benefit is paid and the price of the future claims is re-calculated for the in-force policies. The integrand in (5.4) is the sum at risk to which the insurer is exposed in the event of the policyholder's death. The integrand in (5.4) can also be interpreted as the super-replication strategy for the insurance risk. This super-replication strategy cannot be applied by the insurer since its cost is too high to bear.

We can prove that the first term in (4.6) converges to

$$\lim_{h\to 0}\frac{\mathbb{E}^{\mathbb{P}}_{t,y,f,k}\big[X^{[t,t+h]}(t+h)e^{-rh}-X^{(t,t+h)}(t)\big]}{h}$$
$$\begin{aligned}= \quad & \varphi^k_t(t,y,f)+\varphi^k_y(t,y,f)yr+\varphi^k_f(t,y,f)f\big(\mu_F-\frac{\mu_Y-r}{\sigma_Y}\sigma_F\rho\big)\\ & +\varphi^k_{yf}(t,y,f)yf\sigma_Y\sigma_F\rho+\frac{1}{2}\varphi^k_{yy}(t,y,f)y^2\sigma_Y^2+\frac{1}{2}\varphi^k_{ff}(t,y,f)f^2\sigma_F^2\\ & +kA(t,y,f)+\big(\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\big)k\lambda(t)-\varphi^k(t,y,f)r. \quad (5.5)\end{aligned}$$

If the variance is chosen as the one-period actuarial risk margin in (4.4), then one can prove that the second term in (4.6) converges to

$$\lim_{h\to 0}\frac{Var^{\mathbb{P}}_{t,y,f,k}\big[X^{[t,t+h]}(t+h)e^{-rh}-X^{[t,t+h]}(t)\big]}{h}$$
$$= \quad \big(\varphi^k_f(t,y,f)\big)^2f^2\sigma_F^2(1-\rho^2)+\big(\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\big)^2k\lambda(t). \quad (5.6)$$

If the standard deviation is chosen as the one-period actuarial risk margin, then we clearly have that

$$\lim_{h\to 0}\sqrt{\frac{Var^{\mathbb{P}}_{t,y,f,k}\big[X^{[t,t+h]}(t+h)e^{-rh}-X^{[t,t+h]}(t)\big]}{h}}$$
$$= \quad \sqrt{\big(\varphi^k_f(t,y,f)\big)^2f^2\sigma_F^2(1-\rho^2)+\big(\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\big)^2k\lambda(t)}. \quad (5.7)$$

As we expected, the continuous-time limits of the one-period variance and standard deviation risk margins in (4.6) depend on the processes which govern the quadratic variations (5.3)-(5.4).

We are now ready to state our key result.

**Theorem 5.1.** *a) Let us consider the one-period valuation operator from Proposition 3.4 and the multi-period valuation operator defined by the backward iterations* (4.1) *of the one-period valuation operator with step $h$. We choose the equivalent martingale measure $\hat{\mathbb{Q}}$ from* (2.7) *for the one-period mean-square hedging problem, and variance or standard deviation under the real-world measure $\mathbb{P}$ as the one-period actuarial risk margin:*

$$RM(\xi)=\frac{1}{2}\gamma Var^{\mathbb{P}}[\xi], \quad \textit{or} \quad RM(\xi)=\frac{1}{2}\gamma\sqrt{h}\sqrt{Var^{\mathbb{P}}[\xi]},$$

*where $\gamma$ denotes a risk aversion coefficient. We investigate the discrete-time pricing equation* (4.6)*.*

*b) Let us consider the PDEs*

$$
\begin{aligned}
&\varphi_t^k(t,y,f)+\varphi_y^k(t,y,f)yr+\varphi_f^k(t,y,f)f\big(\mu_F-\frac{\mu_Y-r}{\sigma_Y}\sigma_F\rho\big)\\
&+\varphi_{yf}^k(t,y,f)yf\sigma_Y\sigma_F\rho+\frac{1}{2}\varphi_{yy}^k(t,y,f)y^2\sigma_Y^2+\frac{1}{2}\varphi_{ff}^k(t,y,f)f^2\sigma_F^2\\
&+kA(t,y,f)+\big(\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\big)k\lambda(t)-\varphi^k(t,y,f)r\\
&+\Phi^k\Big(t,\varphi_f^k(t,y,f)f\sigma_F\sqrt{1-\rho^2},\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\Big)=0,\\
&\qquad (t,y,f)\in[0,T)\times(0,\infty)\times(0,\infty),\\
&\varphi^k(T,y,f)=kS(y,f),\quad (y,f)\in(0,\infty)\times(0,\infty),
\end{aligned}
\tag{5.8}
$$

*for* $k\in\{0,...,n\}$*, where* $\Phi^k(t,x_1,x_2)=\frac{1}{2}\gamma(x_1^2+x_2^2k\lambda(t))$ *for the variance risk margin and* $\Phi^k(t,x_1,x_2)=\frac{1}{2}\gamma\sqrt{x_1^2+x_2^2k\lambda(t)}$ *for the standard deviation risk margin.*
*c) We assume that there exist unique solutions* $(\varphi^k)_{k=0,\dots,n}$ *to the PDEs* (5.8) *such that* $\varphi^k\in\mathcal{C}^{1,2,2}([0,T)\times(0,\infty)\times(0,\infty))\cup\mathcal{C}([0,T]\times(0,\infty)\times(0,\infty))$ *and the mixed derivatives* $\varphi_{tx}^k,\varphi_{tf}^k\in\mathcal{C}([0,T)\times(0,\infty)\times(0,\infty))$*, for each* $k\in\{0,...,n\}$*. Moreover, we assume the growth conditions*

$$
\begin{aligned}
|\varphi^k(t,y,f)| &\le K(1+|y|^p+|f|^p),\quad for\quad some\quad p\ge 1,\\
|\varphi_y^k(t,y,f)|+|\varphi_f^k(t,y,f)| &\le K(1+|y|^p+|f|^p),\quad for\quad some\quad p\ge 1,
\end{aligned}
$$

*Under a)-c), the continuous-time valuation operator* $\varphi:=(\varphi^k)_{k=0,\dots,n}$ *determined by the PDEs* (5.8) *satisfies the continuous-time limit of the discrete-time pricing equation* (4.6) *as* $h\to 0$.

**Remark 5.1.** *a) As pointed out by Pelsser & Ghalehjooghi (2016), if we use the standard deviation risk margin, then we must use* $\sqrt{h}\sqrt{Var^{\mathbb{P}}[\xi]}$ *in order to have the convergence for* $h\to 0$.
*b) As far as the smoothness and growth conditions for* $\varphi^k$ *are concerned, we refer to the remark after Proposition 3.1. We point out that we refrain from making any growth assumptions on the second order derivatives as pointed out by Cont & Voltchkova (2005).*

We now discuss two crucial properties of the valuation operator $\varphi$ determined by the PDEs (5.8).

**Theorem 5.2.** *Let us consider the continuous-time valuation operator* $\varphi$ *from Theorem 5.1.*
*(i) The valuation operator has the representation:*

$$
\begin{aligned}
\varphi^k(t,y,f) &= \mathbb{E}_{t,y,f,k}^{\hat{\mathbb{Q}}}\Big[\int_t^T e^{-r(s-t)}dB(s)+\int_t^T e^{-r(s-t)}\Phi(s)ds\Big],\\
&\qquad (t,y,f)\in[0,T]\times(0,\infty)\times(0,\infty),\ k\in\{0,...,n\},
\end{aligned}
\tag{5.9}
$$

*where $\Phi(s)$ is a shorthand notation for*

$$\Phi^{J(s)}\Big(s, \varphi_f^{J(s)}(s, Y(s), F(s))F(s)\sigma_F\sqrt{1-\rho^2},$$
$$\varphi^{J(s)-1}(s, Y(s), F(s)) + D(s, Y(s), F(s)) - \varphi^{J(s)}(s, Y(s), F(s))\Big),$$

*(ii) The valuation operator is market-consistent and actuarial, hence it is fair.*

We now focus on (5.9). If we look at the proof that leads to the PDEs (5.8), in particular, if we look at the limits (5.6)-(5.7), then we can conclude that the term $\Phi$ in the PDEs (5.8) is the continuous-time limit of the one-period actuarial risk margin $RM$. We can call $\Phi$ an *instantaneous actuarial risk margin*, and $\int \Phi(s)$ is the integrated instantaneous actuarial risk margin. The expected value of the integrated instantaneous actuarial risk margin is called the *total actuarial risk margin*. When the valuation operator $\varphi$ is applied to an orthogonal process $B^O$, then the operator $\Phi$ becomes independent of the tradeable risky asset $Y$ and satisfies the measurability assumption (2.13) for an actuarial valuation operator. This property justifies its name as an instantaneous actuarial risk margin, see also point *(ii)* in Theorem 5.2.

The representation (5.9) says that our continuous-time valuation operator $\varphi$ values liabilities as the best estimate of the liability plus the total actuarial risk margin for the liability:

$$\begin{aligned}\varphi_B &= Fair\ Value\ of\ B\\ &= Best\ Estimate\ of\ B\ +\ Total\ Actuarial\ Risk\ Margin\ for\ B.\end{aligned}$$

The best estimate of a liability is the expected value of the future claims from the liability, where the expected value is taken under the measure $\hat{\mathbb{Q}}$ given by (2.7). Let us recall that the equivalent martingale measure $\hat{\mathbb{Q}}$ results from choosing $\zeta(t) = \chi(t) = 0$ in (2.6). It agrees with intuition that the best estimate assumptions for the non-tradeable financial and insurance risks should not include any risk premiums for these risks, see also Happ et al. (2015). We can observe that the best estimate of a liability contingent on the hedgeable financial risk coincides with the market cost of the self-financing investment portfolio which perfectly replicates the claims. The best estimate of a liability contingent on the independent, non-hedgeable financial and insurance risks is the expected cost, under the real-world measure $\mathbb{P}$, of the non-hedgeable claims to be paid. Intuitively, the best estimate of a liability contingent on the hedgeable and non-hedgeable financial and insurance risks consists of the market cost of the self-financing investment portfolio used for replicating the hedgeable part of the claims and the expected, real-world, cost of the non-hedgeable claims left after the application of the hedging portfolio. The total actuarial risk margin, or the integrated instantaneous actuarial risk margin, gives us the capital which the insurer will need to set aside during the duration of the insurance portfolio in order to cover the non-hedgeable financial and insurance risks in adverse scenarios. In contrast, the one-period actuarial risk margin, from which we start in (4.1), provides the capital which the insurer needs to set aside for the next period of length $h$ in order to cover the non-hedgeable risks in adverse scenarios.

The calculations leading to Theorem 5.1 allow us to define the hedging strategy which underlies the continuous-time limit of the multi-period iterated valuation operator (4.1).

**Theorem 5.3.** *Let us consider the continuous-time valuation operator $\varphi$ from Theorem 5.1. The self-financing dynamic hedging strategy which underlies the valuation $\varphi$ is given by*

$$\vartheta^*(t) = \varphi_y^{J(t-)}(t, Y(t), F(t))Y(t) + \varphi_f^{J(t-)}(t, Y(t), F(t))F(t)\frac{\sigma_F}{\sigma_Y}\rho, \quad 0 \leq t \leq T.$$

*The hedging strategy $\vartheta^*$ is market-consistent and actuarial, hence it is fair.*

Let us derive the dynamics of the net asset value in the whole period $[0, T]$ during which the available assets are continuously rebalanced with the hedging strategy $\vartheta^*$ and the liabilities are continuously priced with the valuation operator $\varphi$. Let $X(s) = \varphi(s) - V^*(s)$, where $\varphi$ denotes the valuation operator defined in Theorem 5.1 and $V^*$ is the self-financing hedging portfolio given by (3.3) under the strategy $\vartheta^*$ from Theorem 5.3. From (5.2) and (5.8) we can deduce the dynamics of $X(s)$:

$$\begin{aligned} dX(s) &= X(s)rds - \Phi(s)ds + \varphi_f^{J(s-)}(s, Y(s), F(s))F(s)\sigma_F\sqrt{1-\rho^2}dW_2(s) \\ &\quad + \Big(\varphi^{J(s-)-1}(s, Y(s), F(s)) \\ &\qquad + D(s, Y(s), F(s)) - \varphi^{J(s-)}(s, Y(s), F(s))\Big)d\tilde{N}(s), \quad 0 \leq s \leq T. \end{aligned} \tag{5.10}$$

Let us recall that $NAV(t) = -X(t)$. By the martingale property of the stochastic integrals we find

$$\mathbb{E}^{\mathbb{P}}[NAV(t)e^{-r(t-s)}|\mathcal{F}_s] = NAV(s) + \mathbb{E}^{\mathbb{P}}\Big[\int_s^t e^{-r(u-s)}\Phi(u)du|\mathcal{F}_s\Big], \quad 0 \leq s \leq t \leq T. \tag{5.11}$$

At each time $t \in [0, T)$, the insurer must hold additional capital (the instantaneous actuarial risk margin) determined by $\Phi(t)$ which protects the insurer against adverse scenarios in the evolution of the non-hedgeable claims $dB(t)$ and the non-hedgeable change in the value of the claims $d\varphi(t)$ in an infinitesimal period of time $dt$. The evolution of the non-hedgeable risks is described with the two stochastic integrals in (5.10), see also (5.3)-(5.4). These two stochastic integrals describe the risk that the value of the benefit payments changes due to a change in the non-hedgeable, independent component of the risky asset $F$ and the risk that in the case of the non-hedgeable, independent event of the policyholder's death the insurer pays the death benefit and recalculates the value of the benefit payments for the in-force policies. Please note that the instantaneous actuarial risk margin $\Phi$ offsets the differentials of the stochastic integrals for the non-hedgeable risks in (5.10). At time $t = 0$ the expected cost (i.e. the best estimate) of providing the additional capital $\Phi$ till maturity of the insurance portfolio is equal to $\mathbb{E}^{\hat{\mathbb{Q}}}[\int_0^T e^{-rs}\Phi(s)ds]$ and is a part of the technical provision (5.9) – the total actuarial risk margin. As time $t$ goes by, the technical provision (the value of the benefit stream), the best estimate of the liability and the cost of financing the future instantaneous actuarial risk margins are recalculated. From (5.11) we see that the insurer earns, on average, a risk-free rate on the net asset value and the instantaneous actuarial risk margins accumulated with the risk-free rate. The instantaneous actuarial risk margins are released from the technical provision (5.9) as time passes and, on average, they are not used to cover the losses since the realized loss on the hedgeable risk is always zero and the expected loss on the non-hedgeable risks is also zero, both under $\mathbb{P}$ and $\hat{\mathbb{Q}}$ (the expected value of the stochastic integrals in (5.10) is zero). The insurer applies the investment strategy (5.10), which is in fact the optimal hedging strategy in the mean-square sense for the benefits to be paid in the future and the instantaneous actuarial risk margins to be kept in the future, see Proposition 3.1. If the insurer is exposed to the non-hedgeable financial risk $F$ then the instantaneous risk margin $\Phi$ depends on $F$ and the future required solvency capital can be partially hedged by investing in the risky asset $Y$, if $F$ is correlated with $Y$. Consequently, if the insurer is exposed to the non-hedgeable financial risk correlated with the tradeable risky asset, a part of the total actuarial risk margin included in the fair price $\varphi$ of the benefit stream is invested in the risky asset $Y$. Only if the insurer is exposed to the orthogonal benefit stream $B^O$, the total actuarial risk margin is completely invested in the risk-free bank account. These interpretations in the continuous-time model agree with the interpretations we presented in Section 4 in the discrete-time, multi-period model.

Let us finally compare our results with the results from Pelsser (2010) and Pelsser & Ghalehjooghi (2016). These authors investigate the benefit stream

$$B(t) = S(Y(T), F(T))\mathbf{1}\{t = T\}, \tag{5.12}$$

which is a special case of our general benefit stream (2.5). Pelsser (2010) and Pelsser & Ghalehjooghi (2016) propose a completely different one-period valuation operator $\varrho$, called a two-step valuation operator. They define the multi-period valuation operator by the iterations (4.1), take the limit $h \to 0$ and derive a PDE. Interestingly, for the benefit stream (5.12) and variance/standard deviation one-period valuation operator applied to the non-hedgeable risk our system of PDEs (5.8) reduces to a single PDE which agrees with the PDE derived by Pelsser (2010) and Pelsser & Ghalehjooghi (2016). However, our hedging strategy from Theorem 5.3 is different from the hedging strategy postulated by Pelsser (2010) and Pelsser & Ghalehjooghi (2016). Let us point out that the hedging strategy assumed by

Pelsser (2010) and Pelsser & Ghalehjooghi (2016) is a self-financing, delta-hedging, perfect replication strategy for the actuarial value of the claims which is viewed as a financial derivative contingent on the tradeable risky asset $Y$. Our strategy is a self-financing, delta-hedging strategy which minimizes the mean-square hedging error for the claims. For a comparison of the two strategies, we refer to Example 3.3 in Delong et al. (2019).

# 6 Continuous-time valuation operator – beyond quadratic actuarial risk margins

When we look at the structure of the PDE (5.8), the discrete-time pricing equation (4.6) and the formulas (5.5)-(5.7), we can conclude that the last term $\Phi$ in the PDE is the continuous-time limit of the one-period actuarial risk margin $RM$, and the remaining terms are the continuous-time limit of the one-period expected value pricing principle. In the previous section we called $\Phi$ an instantaneous actuarial risk margin. The instantaneous actuarial risk margins $\Phi$ which we derived for the one-period standard deviation and variance risk margins act on $\varphi^k_f(t,y,f)f\sigma_F\sqrt{1-\rho^2}$ and $\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)$. These two terms can be related to the non-hedgeable part of the process $X^{[t,t+h]}$, or to the non-hedgeable part of the net asset value process, see (5.2)-(5.4). Since in the previous section we used quadratic actuarial risk margins to quantify the non-hedgeable risks, it is natural to use the quadratic variation of the process $X^{[t,t+h]}$ to define the instantaneous actuarial risk margin for the non-hedgeable risks. However, in the theory of stochastic processes and in applications, power variations are also used, see Theorem 2.2 in Barndorff-Nielsen et al. (2006) and Theorem 2.2 in Jacod (2008). Consequently, if we decide to quantify the non-hedgeable risks in $L^q$-norm, then we should use $q$-power variations of the process $X^{[t,t+h]}$. We would choose the instantaneous actuarial risk margin

$$\Phi^k(t,x_1,x_2) = \frac{1}{2}\gamma\Big(|x_1|^q + |x_2|^q k\lambda(t)\Big)^{1/q},$$

and apply it to $\varphi^k_f(t,y,f)f\sigma_F\sqrt{1-\rho^2}, \varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)$.

Let us remark that Rosazza Gianin (2006) and Barrieu & Karoui (2005), among others, develop dynamic risk measures which are defined as solutions to backward stochastic differential equations (BSDEs), see also Pelsser & Stadje (2014). In their framework, our $\mathbb{F}$-adapted benefit stream $B$ would be valued at any time $t \in [0,T]$ with the process $\mathcal{Y}$ which is a solution to the BSDE

$$\begin{aligned}\mathcal{Y}(t) &= \int_t^T dB(s) + \int_t^T \Big(g\big(s,\mathcal{Z}_2(s),\mathcal{U}(s)\big) - \frac{\mu_Y - r}{\sigma_Y}\mathcal{Z}_1(s) - \mathcal{Y}(s-)r\Big)ds \\ &\quad - \int_t^T \mathcal{Z}_1(s)dW_1(s) - \int_t^T \mathcal{Z}_2(s)dW_2(s) - \int_t^T \mathcal{U}(s)d\tilde{N}(s), \quad 0 \le t \le T, \end{aligned} \tag{6.1}$$

where $g$ is interpreted as an instantaneous risk measure or a local preference-based pricing rule for the non-hedgeable risks. Consequently, new dynamic risk measures can be constructed with BSDEs and instantaneous risk measures. From the proof of Proposition 3.1, we find that in our Markovian model the solution to the BSDE (6.1) is given by

$$\begin{aligned}
\mathcal{Y}(t) &= \phi^{J(t)}(t, Y(t), F(t)) \\
&= \mathbb{E}^{\hat{\mathbb{Q}}}\Big[\int_t^T e^{-r(s-t)} dB(s) + \int_t^T e^{-r(s-t)} g\big(s, \mathcal{Z}_2(s), \mathcal{U}(s)\big) ds | \mathcal{F}_t\Big], \\
\mathcal{Z}_1(t) &= \phi_y^{J(t-)}(t, Y(t), F(t)) Y(t)\sigma_Y + \phi_f^{J(t-)}(t, Y(t), F(t)) F(t)\sigma_F \rho, \\
\mathcal{Z}_2(t) &= \phi_f^{J(t-)}(t, Y(t), F(t)) F(t)\sigma_F \sqrt{1-\rho^2}, \\
\mathcal{U}(t) &= \phi^{J(t-)-1}(t, Y(t), F(t)) + D(t, Y(t), F(t)) - \phi^{J(t-)}(t, Y(t), F(t)),
\end{aligned}$$

and the dynamic risk measure $\phi$ induced by the BSDE has exactly the same representation as our continuous-time valuation operator (see point *(i)* of Theorem 5.2).

Consequently, we propose to price the benefit stream $B$ with the operator $\varphi$ which satisfies the system of PDEs:

$$\begin{aligned}
&\varphi_t^k(t,y,f) + \varphi_y^k(t,y,f) yr + \varphi_f^k(t,y,f) f\big(\mu_F - \frac{\mu_Y - r}{\sigma_Y}\sigma_F\rho\big) \\
&+\varphi_{yf}^k(t,y,f) yf\sigma_Y\sigma_F\rho + \frac{1}{2}\varphi_{yy}^k(t,y,f) y^2\sigma_Y^2 + \frac{1}{2}\varphi_{ff}^k(t,y,f) f^2\sigma_F^2 \\
&+kA(t,y,f) + \big(\varphi^{k-1}(t,y,f) + D(t,y,f) - \varphi^k(t,y,f)\big)k\lambda(t) - \varphi^k(t,y,f)r \\
&+\Phi^k\Big(\varphi_f^k(t,y,f) f\sigma_F\sqrt{1-\rho^2}, \varphi^{k-1}(t,y,f) + D(t,y,f) - \varphi^k(t,y,f)\Big) = 0, \\
&\qquad (t,y,f) \in [0,T) \times (0,\infty) \times (0,\infty), \\
&\varphi^k(T,y,f) = kS(y,f), \quad (y,f) \in (0,\infty) \times (0,\infty),
\end{aligned} \tag{6.2}$$

for $k \in \{0, ..., n\}$, where $\Phi$ denotes some instantaneous actuarial risk margin applied to the non-hedgeable part of the insurer's net asset value process. The operator $\Phi$ locally quantifies the non-hedgeable risks which are present in the evolution of the insurer's net asset value. The instantaneous risk margin $\Phi$ should be interpreted as the continuous-time limit of the one-period risk margin $RM$ used in the multi-period iterated valuation operator (4.4). For continuous-time limits of some static risk measures we refer to Stadje (2010). Our previous calculations and elaborations justify the PDE (6.2). We can conclude that the valuation operator arising as the continuous-time limit of the multi-period iterated valuation operator (4.4) with the optimal mean-square hedging portfolio and an arbitrary one-period actuarial risk margin should satisfy the PDEs (6.2) with some function $\Phi$.

**Proposition 6.1.** *Let us define the continuous-time valuation operator $\varphi$ with the PDEs (6.2) with an arbitrary function $\Phi : \{0, ..., n\} \times [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto [0,\infty)$ such that $\Phi^k(t,0,0) = 0$*

*and $\Phi$ satisfies the polynomial growth*

$$|\Phi^k(t,x_1,x_2)| \le K(1+|x_1|^p+|x_2|^p), \quad \textit{for some } p \ge 1.$$

*In case all assumptions from Theorem 5.1 are satisfied, we have that the valuation operator $\varphi$ and the self-financing dynamic hedging strategy $\vartheta^*$ satisfy all properties from Theorems 5.2 and 5.3.*

The quadratic variation (standard deviation or variance instantaneous risk margin) penalizes both losses and gains on the net asset value process (5.2). In applications, we might prefer to use asymmetric instantaneous actuarial risk margins $\Phi$ which only penalize losses on the net asset value process. The theory of fair valuation under general one-period risk margin and general instantaneous risk margin is an open research field.

# 7 Conclusion

We have investigated fair (market-consistent and actuarial) valuation of insurance liability cash-flow streams in continuous time. We have derived a partial differential equation for a continuous-time valuation operator and have proved that our valuation operator is actuarial and market-consistent. We have also shown that our continuous-time fair valuation operator has a natural decomposition into the best estimate of the liability and a risk margin. Applications of the results to the benefit streams discussed in Examples 1-2 in Section 2, explicit results for the fair values, the best estimates, the risk margins and the hedging strategies can be found in Delong et al. (2019).

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# A Appendix: Proofs of the results

The expected values are taken under $\mathbb{P}$, unless a different measure is explicitly stated.

**Proof of Proposition 3.1:** Since the claims $A, D, S$ satisfy (A3) and $(Y, F)$ are geometric Brownian motions, we can show that $\mathbb{E}[|B(T)|^q] < \infty$ for all $q \geq 2$. Let $\mathcal{M}^{\zeta,\chi}$ denote the martingale (2.6) which defines $\mathbb{Q}$. From Theorem 3.1 in Kazamaki (2006) and boundedness of $\chi$, we conclude that $\mathbb{E}[|\mathcal{M}^{\zeta,\chi}(T)|^l] < \infty$ for some $l > 1$. Consequently, we can also prove that $\mathbb{E}^{\mathbb{Q}}[|B(T)|^q] < \infty$ for all $q \geq 2$. The property of predictable representation, see e.g. Chapter XIII.2 in He et al. (1992) or Chapter 2.4 in Delong (2013), gives us

$$\begin{aligned}\int_0^T e^{-rt} dB(t) &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-rt} dB(t)\Big] + \int_0^T Z_1(t) dW_1^{\mathbb{Q}}(t) \\ &\quad + \int_0^T Z_2(t) dW_2^{\mathbb{Q}}(t) + \int_0^T U(t) d\tilde{N}^{\mathbb{Q}}(t). \qquad (A.1)\end{aligned}$$

From Theorem 5.1 in El Karoui et al. (1997) we conclude that

$$\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T |Z_1(t)|^2 dt\Big)^{q/2}\Big] < \infty,$$

for all $q \geq 2$. Since the martingale $\mathcal{N}^{\zeta',\chi'}$ which defines the derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{F}_t = \mathcal{N}^{\zeta',\chi'}(t)$ also satisfies $\mathbb{E}[|\mathcal{N}^{\zeta',\chi'}(T)|^l] < \infty$ for some $l > 1$, we can show that the process $Z_1$ is square integrable under $\mathbb{P}$.

By Theorem 10.1.1 in Delong (2013) the optimal hedging strategy is given by $\theta^*_{B(0,T),\triangle B(T)}(t) = \frac{Z_1(t)}{\sigma_Y} e^{rt}$. Recalling the arguments from above and the function $v$ defined in this proposition, we deduce that the process $M(t) = v^{J(t)}(t, Y(t), F(t)) e^{-rt} + \int_0^t e^{-ru} dB(u)$ is a $\mathbb{Q}$-martingale with the representation (A.1). Since $v$ is a smooth function on $[0, T-\epsilon] \times (0,\infty) \times (0,\infty)$, we can apply Itô's formula to the process $M$ on $[0, T-\epsilon]$. The form of the optimal hedging strategy (3.6) on $[0, T-\epsilon]$ can be now found by comparing the stochastic integral with respect to $W_1$ from $M$ with the stochastic integral with respect to $W_1$ from (A.1). We also have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^{T-\epsilon} \Big| v_y^{J(t-)}(t, Y(t), F(t)) Y(t) + v_f^{J(t-)}(t, Y(t), F(t)) F(t) \frac{\sigma_F}{\sigma_Y}\rho\Big|^2 dt\Big] \\ &\leq \mathbb{E}\Big[\int_0^T \Big|\frac{Z_1(t)}{\sigma_Y} e^{rt}\Big|^2 dt\Big] < \infty,\end{aligned}$$

and we can define the strategy (3.6) on $[0, T]$ by the Fatou's lemma and $\epsilon \to 0$. We also conclude that the optimal hedging strategy is square integrable under $\mathbb{P}$ and $\mathbb{Q}$. $\square$

**Proof of Proposition 3.2:** We refer the reader to Chapter 10.2 in Delong (2013) and the proof of Proposition 3.1. $\square$

**Proof of Proposition 3.3:** *(i)* Let us consider $\tilde{B}(t) = B^Y(t) + B(t)$ where $B^Y$ is a hedgeable process and $B$ is an arbitrary process. By the additivity property of the expected value operator, we get $v^k_{\tilde{B}}(t, y, f) = v^k_{B^Y}(t, y, f) + v^k_B(t, y, f)$, where $v$ is defined in Proposition 3.1. Moreover, $v^k_{B^Y}(t, y, f) = v_{B^Y}(t, y)$. By the definition of the optimal hedging strategy $\theta^*_{\tilde{B}}$ from (3.6), we conclude that condition (2.16) is satisfied. By (A4), for an orthogonal process $B^N$ we have $v^k_{B^N}(t, y, f) = v^k_{B^N}(t)$, and for an orthogonal process $B^{F,\perp\!\!\!\perp}$ we have $v^k_{B^{F,\perp\!\!\!\perp}}(t, y, f) = v_{B^{F,\perp\!\!\!\perp}}(t, f)$ and $\rho = 0$. Consequently, we can deduce that $\theta^*_{B^O}(t) = 0$ and condition (2.17) is satisfied.
*(ii)* We use the results from the proof of *(i)*. We note that the optimal hedging strategy from (3.9) is a linear function of the hedging portfolio, with a constant slope and an intercept which is additive with respect to the benefit stream $\tilde{B} = B^Y + B$. From the linear SDE (3.3) describing the dynamics of the hedging portfolio with the additive benefit stream $\tilde{B} = B^Y + B$, we can conclude that $V^*_{\tilde{B}}(t) = V^*_{B^Y}(t) + V^*_B(t)$. By (3.9) we find $\theta^*_{\tilde{B}}(t) = \theta^*_{B^Y}(t) + \theta^*_B(t)$. Since $B^Y$ is hedgeable, $V^*_{B^Y}(T) = v_{B^Y}(T, Y(T))$ and (2.16) is satisfied.
We give an example showing that the optimal hedging strategy (3.9) is not actuarial. Let us assume that the insurer is exposed to the orthogonal process $B^N$ which only includes the constant survival benefits $S = 1$. From Proposition 3.2 we can conclude that $v^k_{B^N}(t, y, f) = v^k_{B^N}(t), \theta^*_{B^N}(0) = 0$ and $V^*_{B^N}(0) = \mathbb{E}^{\hat{\mathbb{Q}}}[e^{-rT}(n - N(T))] = ne^{-rT}e^{-\int_0^T \lambda(s)ds}$. However, at some latter time $t$ we may have $V^*_{B^N}(t) < \mathbb{E}^{\hat{\mathbb{Q}}}[e^{-r(T-t)}(n - N(T))|\mathcal{F}_t] = (n - N(t))e^{-r(T-t)}e^{-\int_t^T \lambda(s)ds)} = v^{J(t)}_{B^N}(t)$, since the insurance risk may evolve differently from the way the insurer expected at time $t = 0$ and the risk-free investment is not sufficient to cover the reserve at time $t$ (less policyholders have died on $[0, t]$ than the insurer expected at time $t = 0$ and $N(t) < n(1 - e^{-\int_0^t \lambda(s)ds}))$. If $\mu_Y > r$, then the insurer can reduce the deficiency in the reserve and can reduce the mean-square hedging error at the terminal time $T$ by investing in the risky asset $Y$. Consequently, the optimal dynamic hedging strategy (3.9) is not actuarial. Similar arguments can be applied if we consider an orthogonal process $B^O$. $\square$

**Proof of Proposition 3.4:** Let us consider $\tilde{B}(t) = B^Y(t) + B(t)$ where $B^Y$ is a hedgeable process and $B$ is an arbitrary process. By Propositions 3.1 and 3.3, we know that $\theta^*_{\tilde{B}}(t) = \theta^*_{B^Y}(t) + \theta^*_B(t)$ and the optimal hedging strategies do not depend on the value of the hedging portfolio. Consequently, from the linear SDE (3.3) describing the dynamics of the hedging portfolio with the additive benefit stream $\tilde{B} = B^Y + B$, we can conclude that $V^*_{\tilde{B}}(t) =$

$V^*_{B^Y}(t)+V^*_B(t)$. Since $B^Y$ can be perfectly replicated with $V^*_{B^Y}$, we have $\triangle\tilde{B}(T)-V^*_{\tilde{B}}(T) = \triangle B(T)-V^*_B(T)$. From (3.1) we deduce

$$\varrho(\tilde{B}) = V^*_{B^Y}(0)+V^*_B(0)+\pi\Big(\big(\triangle B(T)-V^*_B(T)\big)e^{-rT}\Big) = \varrho(B)+V^*_{B^Y}(0).$$

By Proposition 3.1 we find $V^*_{B^Y}(0)=\mathbb{E}^{\hat{\mathbb{Q}}}[\int_0^T e^{-rs}dB^Y(s)]$. If $B(t)=0$, then $\varrho(B)=\pi(0)=0$ by the first condition in (3.10), and $\varrho(\tilde{B})=\varrho(B^Y)=V^*_{B^Y}(0)$. We can conclude that the valuation operator (3.1) satisfies (2.12).
Let $B^O$ denote an orthogonal process. By Propositions 3.1 and 3.3, $\theta^*_{B^O}(t)=0$ and $V^*_{B^O}(T)=V^*_{B^O}(0)e^{rT}-\int_0^T e^{r(T-s)}dB^O(s)+\triangle B^O(T)$. Substituting the formula for $V^*_{B^O}(T)$ into the definition of the valuation operator (3.1), using the second condition in (3.10) and the assumption that $\pi$ is actuarial, we can easily prove that the valuation operator (3.1) satisfies (2.13). □

**Proof of Theorem 5.1:** We study the terms on the left hand side of equation (4.6) and we choose the variance as the one-period actuarial risk margin $RM$. We consider the process $X^{[t,t+h]}$ given by (5.1) where $\varphi$ denotes the continuous-time valuation operator. Let us fix $(t,y,f,k)\in[0,T)\times(0,\infty)\times(0,\infty)\times\{0,...,k\}$. We fix sufficiently small $h<1$ such that $t+h<T$ and we consider the interval $[t,t+h]$. Consequently, we always have $s\in[t,t+h]$.
- Step 1: We collect some results which we will often use in the sequel. We have the following properties under the measures $\mathbb{P}$ and $\hat{\mathbb{Q}}$: the processes $Y$ and $F$ are geometric Brownian motions, the counting process $N$ has intensity $\lambda$ and $N$ is independent of $(Y,F)$. We consider $Y$, but the same results hold for $F$. Let $Y^{t,y}$ denote the process $Y$ which starts at time $t$ from $y$. We have $Y^{t,y}(s)=ye^{\mu(t-s)-\frac{1}{2}\sigma^2(s-t)+\sigma W(s-t)}=ye^{\mu(t-s)}M^{t,1}(s)$, and $M$ is an exponential martingale. By Doob's inequality we find that

$$\begin{aligned}\mathbb{E}\big[\sup_{s\in[t,t+h]}|Y^{t,y}(s)|^q\big] &\leq K|y|^q\mathbb{E}[|M^{t,1}(t+h)|^q]\\ &= K|y|^q e^{\frac{1}{2}(q^2-q)\sigma^2 h}\leq K|y|^q,\quad q\geq 2.\end{aligned}$$

We can also show that

$$\begin{aligned}&\mathbb{E}\big[\sup_{s\in[t,t+h]}|Y^{t,y+\alpha}(s)-Y^{t,y}(s)|^q\big]\leq K|\alpha|^q,\quad q\geq 2,\\ &\mathbb{E}\big[\sup_{s\in[t,t+h]}|Y^{t,y}(s)-y|^q\big]=|y|^q\mathbb{E}\Big[\sup_{s\in[t,t+h]}\Big|e^{\mu(s-t)}\Big(1+\int_t^s M^{t,1}(u)\sigma dW(u)\Big)-1\Big|^q\Big]\\ &\leq\ K|y|^q\mathbb{E}\Big[\sup_{s\in[t,t+h]}\Big\{\Big|e^{\mu(s-t)}-1\Big|^q+\Big|e^{\mu(s-t)}\int_t^s M^{t,1}(u)\sigma dW(u)\Big|^q\Big\}\Big]\\ &\leq\ K|y|^q\Big(h+\mathbb{E}\Big[\Big|\int_t^{t+h}\big|M^{t,1}(u)\sigma\big|^2du\Big|^{q/2}\Big]\Big)\\ &\leq\ K|y|^q h\Big(1+\mathbb{E}\big[\sup_{u\in[t,t+h]}|M^{t,1}(u)|^q\big]\Big)\leq K|y|^q h,\quad q\geq 2,\end{aligned}$$

where we use the Burkholder-Davis-Gundy inequality. The same moment estimates hold under $\hat{\mathbb{Q}}$. We can note that the mapping $(t, y, s) \mapsto Y^{t,y}(s)$ is a.s continuous. Moreover, $\frac{d}{dy}Y^{t,y}(s) = \frac{Y^{t,y}(s)}{y}$. We can also prove that

$$\begin{aligned}
\mathbb{E}\big[\sup_{s\in[t,t+h]} |J^{t,k}(s) - k|^q\big] &= \mathbb{E}\big[|J^{t,k}(t+h) - k|^q\big] \le K\mathbb{E}\big[|J^{t,k}(t+h) - k|\big] \\
&= K\mathbb{E}\Big[\int_t^{t+h} J^{t,k}(u)\lambda(u)du\Big] \le Kh, \quad q \ge 2.
\end{aligned}$$

- Step 2: We investigate the function (4.2). We define

$$\begin{aligned}
v^k(s,y,f) &= \mathbb{E}^{\hat{\mathbb{Q}}}_{s,y,f,k}\Big[\int_s^{t+h} e^{-r(u-s)}dB(u) + e^{-r(t+h-s)}\varphi^{J(t+h)}(t+h, Y(t+h), F(t+h))\Big] \\
&= \mathbb{E}^{\hat{\mathbb{Q}}}\Big[k\int_s^{t+h} e^{-r(u-s)}A(u, Y^{s,y}(u), F^{s,f}(u))e^{-\int_s^u \lambda(z)dz}du \\
&\quad +k\int_s^{t+h} e^{-r(u-s)}D(u, Y^{s,y}(u), F^{s,y}(u))e^{-\int_s^u \lambda(z)dz}\lambda(u)du \\
&\quad +e^{-r(t+h-s)}\sum_{i=0}^{k}\varphi^i(t+h, Y^{s,y}(t+h), F^{s,f}(t+h))Pr(J(t+h) = i|J(t) = k)\Big],
\end{aligned}$$

where we use the independence between $N$ and $(Y, F)$ from Step 1. The function $(s, y, f) \mapsto v^k(s, y, f)$ is finite and continuous on $[t, t+h] \times (0,\infty) \times (0,\infty)$ by Step 1), the growth and continuity conditions for $A, D, \varphi^i$, boundedness of $\lambda$ and the uniform integrability of $(\Lambda^{\tilde{s},\tilde{y},\tilde{f}})_{(\tilde{s},\tilde{y},\tilde{f})\in[s-\epsilon,s+\epsilon]\times[y-\epsilon,y+\epsilon]\times[f-\epsilon,f+\epsilon]}$ where $v^k(s, y, f) = \mathbb{E}^{\hat{\mathbb{Q}}}[\Lambda^{s,y,f}]$, see Lemma 2 from Heath & Schweizer (2000). From Theorem 1 in Heath & Schweizer (2000) we can conclude that $v^k \in \mathcal{C}^{1,2,2}([t, t+h) \times (0,\infty) \times (0,\infty)) \cup \mathcal{C}([t, t+h] \times (0,\infty) \times (0,\infty))$, for each $k \in \{0, ..., n\}$. Consequently, we can apply Proposition 3.1.

- Step 3: The hedging strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$ is given by (4.3). Let us denote $\theta^*_{B(t,t+h),\varphi(t+h)}$ by $\theta^*$. We derive a more explicit formula for $\theta^*$. We will calculate the derivative $v^k_y$. The derivative $v^k_f$ can be calculated in the same way. We consider $\alpha < 1$. Let

$$\begin{aligned}
\nabla A_\alpha(u) &= \frac{A(u, Y^{s,y+\alpha}(u), F(u)) - A(u, Y^{s,y}(u), F(u))}{\alpha}, \\
\nabla\varphi^i_\alpha &= \frac{\varphi^i(t+h, Y^{s,y+\alpha}(t+h), F(t+h)) - \varphi^i(t+h, Y^{s,y}(t+h), F(t+h))|}{\alpha},
\end{aligned}$$

for $u \in [s, t+h]$ and $i \in \{0, ..., n\}$. By the Lipshitz property of $A$ we have the estimate

$$|\nabla A_\alpha(u)| \le K\frac{sup_{u\in[s,t+h]}|Y^{s,y+\alpha}(u) - Y^{s,y}(u)|}{\alpha}.$$

By Fubini's theorem, we have that $\mathbb{E}^{\hat{\mathbb{Q}}}[\int_s^{t+h}\nabla A_\alpha(u)du] = \int_s^{t+h}\mathbb{E}^{\hat{\mathbb{Q}}}[\nabla A_\alpha(u)]du$. We can show that $\sup_{u\in[s,t+h]}\mathbb{E}^{\hat{\mathbb{Q}}}[\nabla A_\alpha(u)] \le \sup_{u\in[s,t+h]}\sqrt{\mathbb{E}^{\hat{\mathbb{Q}}}[|\nabla A_\alpha(u)|^2]} \le K$ by the moment estimates from Step 1. Hence, by the dominated convergence theorem we can take the limit $\alpha \to 0$ under the integral $\int_s^{t+h} \mathbb{E}^{\hat{\mathbb{Q}}}[\nabla A_\alpha(u)]du$. Using again the moment estimates from Step 1, we can deduce that

$$\sup_{\alpha \in [0,1)} \mathbb{E}^{\hat{\mathbb{Q}}}[|\nabla A_\alpha(u)|^2] \leq K,$$

for each fixed $u \in [s, t+h]$. Consequently, by the uniform integrability of $(\nabla A_\alpha(u))_\alpha$ we can take the limit $\alpha \to 0$ under $\mathbb{E}^{\hat{\mathbb{Q}}}[\nabla A_\alpha(u)]$ for each fixed $u \in [s, t+h]$. Recall that a) $z \mapsto A(., z, .)$ is Lipschitz and has a countable number of non-differentiable points, b) the law of $Y^{t,y}(u)$ is absolutely continuous and $y \mapsto Y^{t,y}(u)$ is differentiable by Step 1. Therefore, we can define the derivative

$$\frac{d}{dy} A(u, Y^{s,y}(u), F(u)) = A_y(u, Y^{s,y}(u), F(u)) \frac{Y^{s,y}(u)}{y}, \quad a.s.,$$

for $u \in [s, t+h]$. Analogously, we can define $\frac{d}{dy} D(u, Y^{s,y}(u), F(u))$. By the mean value theorem and the growth conditions for $\varphi^i$, we have the estimate

$$|\nabla \varphi^i_\alpha| \leq K\big(1 + |Y^{s,y+\alpha}(t+h)|^p + |Y^{s,y}(t+h)|^p\big) \frac{sup_{u \in [s,t+h]} |Y^{s,y+\alpha}(u) - Y^{s,y}(u)|}{\alpha}.$$

By the moment estimates from Step 1, the sequence $(\nabla \varphi^i_\alpha)_\alpha$ is uniformly integrable, since $\sup_{\alpha \in [0,1)} \mathbb{E}^{\hat{\mathbb{Q}}}[|\nabla \varphi^i_\alpha|^2] \leq K$. Hence, we can take the limit $\alpha \to 0$ under the expected value $\mathbb{E}^{\hat{\mathbb{Q}}}[\nabla \varphi^i_\alpha]$.

Collecting all the results, we conclude that

$$\begin{aligned}
&\theta^*(t) = \theta^{*,k}(s, y, f) \\
&= \mathbb{E}^{\hat{\mathbb{Q}}}_{s,y,f,k}\Big[\int_s^{t+h} e^{-r(u-s)} A_y(u, Y(u), F(u)) Y(u) J(u) du \\
&\quad + \int_s^{t+h} D_y(u, Y(u), F(u)) Y(u) dN(u) \\
&\quad + e^{-r(t+h-s)} \varphi_y^{J(t+h)}(t+h, Y(t+h), F(t+h)) Y(t+h)\Big] \\
&\quad + \mathbb{E}^{\hat{\mathbb{Q}}}_{s,y,f,k}\Big[\int_s^{t+h} e^{-r(u-s)} A_f(u, Y(u), F(u)) F(u) J(u) du \\
&\quad + \int_s^{t+h} D_f(u, Y(u), F(u)) F(u) dN(u) \\
&\quad + e^{-r(t+h-s)} \varphi_f^{J(t+h)}(t+h, Y(t+h), F(t+h)) F(t+h)\Big] \frac{\sigma_F}{\sigma_Y} \rho, 
\end{aligned} \tag{A.2}$$

where we use the assumption that $(z_1, z_2) \mapsto \varphi^i(t+h, z_1, z_2)$ is continuously differentiable for $t + h < T$.

<u>- Step 4:</u> Let $\vartheta^{*,k}(s, y, f) = \varphi_y^k(s, y, f) y + \varphi_f^k(s, y, f) f \frac{\sigma_F}{\sigma_Y} \rho$. We will derive estimates for $\theta^{*,k}(s, y, f), \vartheta^{*,k}(t, y, f), |\theta^{*,k}(s, y, f) - \vartheta^{*,k}(s, y, f)|$. We will only focus on the terms in $\theta^*$

and $\vartheta^*$ which contain derivatives with respect to $y$. The derivatives with respect to $f$ can be treated in the same way.

From the growth condition for $\varphi^k$ we deduce the first estimate

$$\begin{aligned}|\vartheta^{*,k}(s,y,f)| &\leq K(1+|y|^p+|f|^p)(|y|+|f|) \\ &\leq K(1+\max\{|y|^{p+1},|f|^{p+1}\}) \leq K(1+|y|^{p+1}+|f|^{p+1}).\end{aligned} \tag{A.3}$$

Since $A$ and $D$ are Lipschitz, their derivatives are bounded. Hence, we can conclude that

$$\begin{aligned}&\Big|\int_s^{t+h} e^{-r(u-s)}A_y(u,Y(u),F(u))Y(u)J(u)du+\int_s^{t+h} D_y(u,Y(u),F(u))Y(u)dN(u) \\ &\quad +e^{-r(t+h-s)}\varphi_y^{J(t+h)}(t+h,Y(t+h),F(t+h))Y(t+h)\Big| \\ \leq\ & K\Big(\sup_{u\in[s,t+h]}|Y(u)|h+\sup_{u\in[s,t+h]}|Y(u)||J(t+h)-J(s)| \\ &\quad +1+\sup_{u\in[s,t+h]}|Y(u)|^{p+1}+\sup_{u\in[s,t+h]}|F(u)|^{p+1}\Big), \\ \leq\ & K\Big(1+\sup_{u\in[s,t+h]}|Y(u)|^{p+1}+\sup_{u\in[s,t+h]}|F(u)|^{p+1}\Big)\Big(1+h+\big|J(t+h)-J(s)\big|\Big),\end{aligned} \tag{A.4}$$

where again we use the growth condition for $\varphi^k$ and estimates similar to (A.3). By (A.2), (A.4), the moment estimates from Step 1) and the independence between $(Y,F)$ and $N$, we get the second estimate

$$|\theta^{*,k}(s,y,f)| \leq K(1+|y|^{p+1}+|f|^{p+1})(1+h). \tag{A.5}$$

Let us introduce the following stopping times:

$$\begin{aligned}\mathcal{T}_1 &= \inf\{s\geq t: |Y^{t,y}(s)-y|\geq\epsilon\}, \\ \mathcal{T}_2 &= \inf\{s\geq t: |F^{t,f}(s)-y|\geq\epsilon\}, \\ \mathcal{T}_3 &= \inf\{s\geq t: |J^{t,k}(s)-y|\geq 1\}, \\ \mathcal{T} &= \mathcal{T}_1\wedge\mathcal{T}_2\wedge\mathcal{T}_3,\end{aligned}$$

where we fix sufficiently small $\epsilon<1$ such that $y-\epsilon>0, f-\epsilon>0$ for the pair $(y,f)$ investigated in (4.6). Consequently, $\epsilon$ depends on $(y,f)$. By Chebyshev's inequality and the moment estimates from Step 1, we deduce that

$$\begin{aligned}\mathbb{P}(\mathcal{T}\leq t+h) &\leq \mathbb{P}\big(\sup_{s\in[t,t+h]}|Y^{t,y}(s)-y|\geq\epsilon\big)+\mathbb{P}\big(\sup_{s\in[t,t+h]}|F^{t,f}(s)-y|\geq\epsilon\big) \\ &\quad +\mathbb{P}\big(\sup_{s\in[t,t+h]}|J^{t,k}(s)-k|\geq 1\big) \\ &\leq \frac{\mathbb{E}\big[\sup_{s\in[t,t+h]}|Y^{t,y}(s)-y|^2\big]}{\epsilon^2}+\frac{\mathbb{E}\big[\sup_{s\in[t,t+h]}|F^{t,f}(s)-y|^2\big]}{\epsilon^2} \\ &\quad +\mathbb{E}\big[\sup_{s\in[t,t+h]}|J^{t,k}(s)-k|^2\big]\leq K_{y,f}h,\end{aligned} \tag{A.6}$$

where $K_{y,f}$ denotes a constant which depends on the pair $(y, f)$.
We now investigate the key term which we will use in the proof of the convergence of (4.6). We notice that

$$
\begin{aligned}
&\mathbb{E}_{t,y,f,k}\Big[\int_t^{t+h}\Big|\theta^{*,J(s)}(s,Y(s),F(s))-\vartheta^{*,J(s)}(s,Y(s),F(s))\Big|ds\Big]\\
\leq\;& \mathbb{E}_{t,y,f,k}\Big[\int_t^{t+h}\Big|\int_s^{t+h}e^{-r(u-s)}A_y(u,Y(u),F(u))Y(u)J(u)du\\
&+\int_s^{t+h}D_y(u,Y(u),F(u))Y(u)dN(u)\Big|ds\\
&+\int_t^{t+h}e^{-r(t+h-s)}\Big|\varphi_y^{J(t+h)}(t+h,Y(t+h),F(t+h))Y(t+h)\\
&\quad-\varphi_y^{J(s)}(s,Y(s),F(s))Y(s)\Big|ds\Big]+\ \textit{the terms with } f-\textit{derivatives}\\
=\;& \mathbb{E}_{t,y,f,k}\Big[H\mathbf{1}\{\mathcal{T}\leq t+h\}+H\mathbf{1}\{\mathcal{T}>t+h\}\Big],
\end{aligned}
\tag{A.7}
$$

where we use the optimal hedging strategy (A.2), Fubini's theorem and the property of conditional expectations. By (A.4), the growth condition for $\varphi^k$, the moment estimates from Step 1 and (A.6), we derive

$$
\begin{aligned}
&\mathbb{E}_{t,y,f,k}\Big[H\mathbf{1}\{\mathcal{T}\leq t+h\}\Big]\\
\leq\;& K\mathbb{E}_{t,y,f,k}\Big[\Big(1+\sup_{u\in[t,t+h]}|Y(u)|^{p+1}+\sup_{u\in[t,t+h]}|F(u)|^{p+1}\Big)\mathbf{1}\{\mathcal{T}\leq t+h\}\Big]h\\
\leq\;& K\sqrt{\mathbb{E}_{t,y,f,k}\Big[1+\sup_{u\in[t,t+h]}|Y(u)|^{2p+2}+\sup_{u\in[t,t+h]}|F(u)|^{2p+2}\Big]}\sqrt{\mathbb{P}\big(\mathcal{T}\leq t+h\big)}h\\
\leq\;& K_{y,f}(1+|y|^{2p+2}+|f|^{2p+2})\sqrt{h}h=K_{y,f}h^{3/2}.
\end{aligned}
\tag{A.8}
$$

If $\mathcal{T}>t+h$, then $J(s)=k$ (or equivalently $dN(s)=0$), $Y(s)\in[y-\epsilon,y+\epsilon]$, $F(s)\in[f-\epsilon,f+\epsilon]$ for $s\in[t,t+h]$. Since $\varphi^k$ is continuously differentiable in $[0,T)\times(0,\infty)\times(0,\infty)$ by our assumption, then we can conclude that the derivatives of the function $\varphi_y^k(t,y,f)$ with respect to $t$, $y$ and $f$ are bounded on $[t,t+h]\times[y-\epsilon,y+\epsilon]\times[f-\epsilon,f+\epsilon]$ by a constant $K_{t,y,f}$ which depends on $(t,y,f)$. We point out that this constant is not affected when we take the limit $h\to 0$. To be more precise, first we fix $h_0$ such that $t+h_0<T$, next we find a global constant $K_{t,y,f}$ on $[t,t+h_0]\times[y-\epsilon,y+\epsilon]\times[f-\epsilon,f+\epsilon]$, and finally we take the limit $h\to 0, h\leq h_0$ in (4.6). By the mean value theorem and the growth conditions for $\varphi^k$,

we can conclude that

$$
\begin{aligned}
&\Big|\varphi_y^{J(t+h)}(t+h, Y(t+h), F(t+h))Y(t+h) - \varphi_y^{J(s)}(s, Y(s), F(s))Y(s)\Big|\mathbf{1}\{\mathcal{T} > t+h\} \\
\le\ & \Big\{\Big|\varphi_y^k(t+h, Y(t+h), F(t+h))Y(t+h) - \varphi_y^k(t+h, Y(t+h), F(t+h))Y(s)\Big| \\
&+\Big|\varphi_y^k(t+h, Y(t+h), F(t+h))Y(s) - \varphi_y^k(s, Y(t+h), F(t+h))Y(s)\Big| \\
&+\Big|\varphi_y^k(s, Y(t+h), F(t+h))Y(s) - \varphi_y^k(s, Y(s), F(t+h))Y(s)\Big| \\
&+\Big|\varphi_y^k(s, Y(s), F(t+k))Y(s) - \varphi_y^k(Y(s), Y(s), F(s))Y(s)\Big|\Big\}\mathbf{1}\{\mathcal{T} > t+h\} \\
\le\ & \Big\{K\big(1 + |Y(t+h)|^p + |F(t+h)|^p\big)|Y(t+h) - Y(s)| + K_{t,y,f}Y(s)h \\
&+K_{t,y,f}|Y(t+h) - Y(s)|Y(s) + K_{t,y,f}|F(t+h) - F(s)|Y(s)\Big\}\mathbf{1}\{\mathcal{T} > t+h\} \\
\le\ & K_{t,y,f}\Big(1 + \sup_{u\in[t,t+h]}|Y(u)|^p + \sup_{u\in[t,t+h]}|F(u)|^p\Big) \\
&\times\Big(h + \sup_{u\in[t,t+h]}|Y(t+h) - Y(u)| + \sup_{u\in[t,t+h]}|F(t+h) - F(u)|\Big) \\
\le\ & K_{t,y,f}\Big(1 + \sup_{u\in[t,t+h]}|Y(u)|^p + \sup_{u\in[t,t+h]}|F(u)|^p\Big) \\
&\times\Big(h + \sup_{u\in[t,t+h]}|Y(u) - Y(t)| + \sup_{u\in[t,t+h]}|F(u) - F(t)|\Big).
\end{aligned}
$$

Combining the above estimate with

$$
\begin{aligned}
&\Big|\int_s^{t+h} e^{-r(u-s)}A_y(u, Y(u), F(u))Y(u)J(u)du \\
&\quad+\int_s^{t+h} D_y(u, Y(u), F(u))Y(u)dN(u)\Big|\mathbf{1}\{\mathcal{T} > t+h\} \le K\sup_{u\in[t,t+h]}|Y(u)|h,
\end{aligned}
$$

and using the moment estimates from Step 1, we find

$$\mathbb{E}_{t,y,f,k}\Big[H\mathbf{1}\{\mathcal{T} > t+h\}\Big] \le K_{t,y,f}h^{3/2}. \tag{A.9}$$

Collecting (A.7), (A.8) and (A.9), we establish the convergence

$$\lim_{h\to 0}\mathbb{E}_{t,y,f,k}\Big[\frac{1}{h}\int_t^{t+h}\Big|\theta^{*,J(s)}(s, Y(s), F(s)) - \vartheta^{*,J(s)}(s, Y(s), F(s))\Big|ds\Big] = 0. \tag{A.10}$$

By the dominated convergence theorem (justified by (A.3) and the moment estimates from Step 1) and the differentiability of the Lebesgue integral, we find that

$$\lim_{h\to 0}\mathbb{E}_{t,y,f,k}\Big[\frac{1}{h}\int_t^{t+h}\vartheta^{*,J(s)}(s, Y(s), F(s))ds\Big] = \varphi_y^k(t,y,f)y + \varphi_f^k(t,y,f)f\frac{\sigma_F}{\sigma_Y}\rho. \tag{A.11}$$

- Step 5: We can now take the limits of the terms on the left hand side of equation (4.6). We divide the equation (4.6) by $h$. We deal with the first term in (4.6). The optimal hedging strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$ is given by (A.2), and the dynamics of the hedging portfolio is given by (3.3). Since $\varphi^k \in \mathcal{C}^{1,2,2}([t,t+h]\times(0,\infty)\times(0,\infty))$ by our assumption, then we can apply Itô's formula to derive the dynamics of $X^{[t,t+h]}$. The dynamics of $X^{[t,t+h]}$ on $[t,t+h]$ is given by (5.2). We have

$$\begin{aligned}
&\frac{\mathbb{E}_{t,y,f,k}\big[X^{[t,t+h]}(t+h)e^{-rh} - X^{[t,t+h]}(t)\big]}{h} \\
&= \frac{\mathbb{E}_{t,y,f,k}\big[X^{[t,t+h]}(t+h)e^{-r(t+h)} - X^{[t,t+h]}(t)e^{-rt}\big]}{h}e^{rt} \\
&= \frac{\mathbb{E}_{t,y,f,k}\big[\int_t^{t+h}\big(-re^{-rs}X^{[t,t+h]}(s)ds + e^{-rs}dX^{[t,t+h]}(s)\big]}{h}e^{rt}. \qquad (A.12)
\end{aligned}$$

From the proof of Proposition 3.1 we can conclude that the optimal hedging strategy $\theta^*_{B(t,t+h),\varphi(t+h)}$ is square integrable on $[t,t+h]$. This square integrability, together with the growth conditions for $D, \varphi^k$ and the moment estimates from Step 1, allows us to conclude that the three stochastic integrals in (5.2) are martingales with expected values equal to zero. We can write (A.12) as

$$\begin{aligned}
&\frac{\mathbb{E}_{t,y,f,k}\big[-\int_t^{t+h} e^{-rs}\theta^{*,J(s)}(s,Y(s),F(s))(\mu_Y - r)ds\big]}{h}e^{rt} \\
&+\frac{\mathbb{E}_{t,y,f,k}\big[\int_t^{t+h} e^{-rs}\Psi^{J(s)}(s,Y(s),F(s))ds\big]}{h}e^{rt}, \qquad (A.13)
\end{aligned}$$

where $\Psi$ collects all the remaining terms. We use (A.10) and (A.11), and we conclude that the limit $h \to 0$ of the first term in (A.13) is equal to $-\vartheta^{*,k}(t,y,f)(\mu_Y - r)$. Since the function $\varphi^k$ satisfies the PDE (5.8), we have

$$\begin{aligned}
\Psi^k(s,y,f) &= \varphi^k_y(s,y,f)y(\mu - r) + \varphi^k_f(s,y,f)f\frac{\mu_Y - r}{\sigma_Y}\sigma_F\rho \\
&\quad -\Phi^k\Big(\varphi^k_f(s,y,f)f\sigma_F\sqrt{1-\rho^2}, \varphi^{k-1}(s,y,f) + D(s,y,f) - \varphi^k(s,y,f)\Big),
\end{aligned}$$

and, using the growth condition for $\varphi^k$, we can derive the estimate

$$|\Psi^k(s,y,f)| \le K\big(1 + |y|^{2p+2} + |f|^{2p+2}\big),$$

By the dominated convergence theorem (justified by (A.14) and the moment estimates from Step 1) and the differentiability of the Lebesgue integral, we can derive that the limit for $h \to 0$ of the second term in (A.13) is equal to $\Psi^k(t,y,f)$. The limit (5.5) is proved.

- Step 6: We now deal with the second term in (4.6). It is clear that

$$Var[\xi] = \big(E[(\xi e^{-rt})^2] - (\mathbb{E}[\xi e^{-rt}])^2\big)e^{2rt}. \qquad (A.14)$$

We start with the second term in (A.14). Using the dynamics (5.2), equations (A.12)-(A.13) and applying the Burkholder-Davis-Gundy inequality, we can derive

$$
\begin{aligned}
&\mathbb{E}_{t,y,f,k}\Big[\sup_{s\in[t,t+h]}\big|X^{[t,t+h]}(s)e^{-rs}-X^{[t,t+h]}(t)e^{-rt}\big|^q\Big] \\
&= \mathbb{E}_{t,y,f,k}\Big[\sup_{s\in[t,t+h]}\Big|-\int_t^s e^{-ru}\theta^{*,J(u-)}(u,Y(u),F(u))(\mu_Y-r)du \\
&\quad+\int_t^s e^{-ru}\Psi^{J(u-)}(u,Y(u),F(u))du \\
&\quad+\int_t^s e^{-ru}\Big(\theta^{*,J(u-)}(u,Y(u),F(u))-\vartheta^{*,J(u-)}(u,Y(u),F(u))\Big)\sigma_Y dW_1(u) \\
&\quad+\int_t^s e^{-ru}\varphi_f^{J(u-)}(u,Y(u),F(u))F(u)\sigma_F\sqrt{1-\rho^2}dW_2(u), \\
&\quad+\int_t^s e^{-ru}\Big(\varphi^{J(u-)-1}(u,Y(u),F(u))+D(u,Y(u),F(u))-\varphi^{J(u-)}(u,Y(u),F(u))\Big)d\tilde{N}(u)\Big|^q\Big] \\
&\le K\mathbb{E}_{t,y,f,k}\Big[h\int_t^{t+h}\big|\theta^{*,J(u)}(u,Y(u),F(u))\big|^q du+h\int_t^{t+h}\big|\Psi^{J(u)}(u,Y(u),F(u))\big|^q du \\
&\quad+\Big|\int_t^{t+h}\big|\theta^{*,J(u)}(u,Y(u),F(u))-\vartheta^{*,J(u)}(u,Y(u),F(u))\big|^2du\Big|^{q/2} \\
&\quad+\Big|\int_t^{t+h}|\varphi_f^{J(u)}(u,Y(u),F(u))F(u)|^2du\Big|^{q/2} \\
&\quad+\Big|\int_t^{t+h}\big|\varphi^{J(u)-1}(u,Y(u),F(u))+D(u,Y(u),F(u))-\varphi^{J(u)}(u,Y(u),F(u))\big|^2dN(u)\Big|^{q/2}\Big],
\end{aligned}
$$

for any $q\ge 2$. By the growth conditions for $A, D, \varphi^k, \Psi^k$, the moment estimates from Step 1) and the estimates (A.3), (A.5), we can conclude that

$$
\mathbb{E}_{t,y,f,k}\big[\sup_{s\in[t,t+h]}\big|X^{[t,t+h]}(s)e^{-rs}-X^{[t,t+h]}(t)e^{-rt}\big|^q\big]\le K_{y,f}h,\quad q\ge 2. \tag{A.15}
$$

The estimate (A.15) and the limit (5.5) now yield

$$
\lim_{h\to 0}\frac{\big(\mathbb{E}_{t,y,f,k}\big[X^{[t,t+h]}(t+h)e^{-r(t+h)}-X^{[t,t+h]}(t)e^{-rt}\big]\big)^2}{h}e^{2rt}=0. \tag{A.16}
$$

We investigate the first term in (A.14):

$$
\begin{aligned}
&\mathbb{E}_{t,y,f,k}\big[\big|X^{[t,t+h]}(t+h)e^{-r(t+h)}-X^{[t,t+h]}(t)e^{-rt}\big|^2\big] \\
&= \mathbb{E}_{t,y,f,k}\big[\big|X^{[t,t+h]}(t+h)e^{-r(t+h)}\big|^2 \\
&\quad-2X^{[t,t+h]}(t+h)e^{-r(t+h)}X^{[t,t+h]}(t)e^{-rt}+\big|X^{[t,t+h]}(t)e^{-rt}\big|^2\big] \\
&= 2\mathbb{E}_{t,y,f,k}\Big[\int_t^{t+h}\Big(X^{[t,t+h]}(s)e^{-rs}-X^{[t,t+h]}(t)e^{-rt}\Big)\Big(-re^{-rs}X^{[t,t+h]}(s)ds+e^{-rs}dX^{[t,t+h]}(s)\Big)\Big] \\
&\quad+\mathbb{E}_{t,y,f,k}\Big[\int_t^{t+h}e^{-2rs}d[X^{[t,t+h]},X^{t,t+h]}](s)\Big]:=2E_1^h+E_2^h, 
\end{aligned}\tag{A.17}
$$

where we apply Itô formula to $|X^{[t,t+h]}(s)e^{-rs}|^2$ and use the dynamics of the discounted process $X^{[t,t+h]}(s)e^{-rs}$.

First, we consider $E_1^h$. By (A.5) and the moment estimates from Step 1), we can prove that $\mathbb{E}_{t,y,f,k}[\int_t^{t+h}|\theta^{*,J(s)}(s,Y(s),F(s))|^q ds]<\infty$ for any $q\geq 2$. This integrability condition together with the growth conditions for $D,\varphi^k$ and estimate (A.15) yield that the stochastic integrals with respect to Brownian motions and the compensated counting process which we have in $E_1^h$, see (5.2), are square integrable martingales and their expected values are equal to zero. Using (A.13) we have

$$
\begin{aligned}
|E_1^h| \leq & \sqrt{\mathbb{E}_{t,y,f,k}\big[\sup_{s\in[t,t+h]}\big|X^{[t,t+h]}(s)e^{-rs}-X^{[t,t+h]}(t)e^{-rt}\big|^2\big]}\\
&\times\sqrt{\mathbb{E}_{t,y,f,k}\big[\sup_{s\in[t,t+h]}|\theta^{*,J(s)}(s,Y(s),F(s))|^2+\sup_{s\in[t,t+h]}|\Psi^{J(s)}(s,Y(s),F(s))|^2\big]h}\\
\leq\ & K_{y,f}\sqrt{h}h,
\end{aligned}
$$

where in the last line we use the estimates (A.5),(A.15), (A.14) and the moment estimates from Step 1). Hence, we conclude that

$$
\lim_{h\to 0}\frac{1}{h}E_1^h=0. \tag{A.18}
$$

We finally deal with $E_2^h$. We can calculate the quadratic variation and we find that

$$
\begin{aligned}
E_2^h=\mathbb{E}_{t,y,f,k}\Big[\int_t^{t+h} & e^{-2rs}\Big(\big|\theta^{*,J(s)}(s,Y(s),F(s))-\vartheta^{*,J(s)}(s,Y(s),F(s))\big|^2\\
+&\big|\varphi_f^{J(s)}(s,Y(s),F(s))\big|^2F^2(s)\sigma_F^2(1-\rho^2)\\
+&\big|\varphi^{J(s)-1}(s,Y(s),F(s))+D(s,Y(s),F(s))-\varphi^{J(s)}(s,Y(s),F(s))\big|^2J(s)\lambda(s)\Big)ds\Big].
\end{aligned}
$$

As in (A.10), we can deduce that

$$
\lim_{h\to 0}\mathbb{E}_{t,y,f,k}\Big[\frac{1}{h}\int_t^{t+h}\big|\theta^{*,J(s)}(s,Y(s),F(s))-\vartheta^{*,J(s)}(s,Y(s),F(s))\big|^2ds\Big]=0.
$$

By the dominated convergence theorem and the differentiability of the Lebesgue integral, we derive the limit:

$$
\begin{aligned}
\lim_{h\to 0}\frac{1}{h}E_2h &= e^{-2rt}\Big(\big(\varphi_f^k(t,y,f)\big)^2f^2\sigma_F^2(1-\rho^2)\\
&\quad+\big(\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\big)^2k\lambda(t)\Big).
\end{aligned} \tag{A.19}
$$

We collect (A.14), (A.16)-(A.19) and we get the desired limit (5.6). □

**Proof of Theorem 5.2:** *(i)* The representation (5.9) is a Feynman-Kac formula for the PDEs (5.8). The processes $(Y,F)$ are geometric Brownian motions under $\hat{\mathbb{Q}}$, the process

$N$ has the same intensity $\lambda$ under $\hat{\mathbb{Q}}$. By the growth conditions for $A, D, S, \varphi^k, \Phi^k$ and the moment estimates from Step 1) in the proof of Theorem 5.1, we can show that the random variable inside the expected value in (5.9) has finite moments of all orders under $\hat{\mathbb{Q}}$. Consequently, the process $\varphi^{J(t)}(t, Y(t), F(t))e^{-rt} + \int_0^t e^{-rs}dB(s) + \int_0^t e^{-rs}\Phi(s)ds$ is a $\hat{\mathbb{Q}}$-martingale. Since $\varphi^k$ is smooth, we can apply Itô's formula and the drift term of the martingale under $\hat{\mathbb{Q}}$ must be zero, see also the proof of Proposition 3.1.

*(ii)* Let us consider $\tilde{B}(t) = B^Y(t) + B(t)$ where $B^Y$ is a hedgeable process and $B$ is an arbitrary process. Let $\tilde{\varphi}$ denotes the valuation operator which satisfies the PDEs (5.8) for the benefit stream $\tilde{B}$. Similarly, $\varphi^Y$ and $\varphi$ are the valuation operators for $B^Y$ and $B$, respectively. Since $B^Y$ only depends on $Y$, we expect that $\varphi^{Y,k}(t, y, f) = \varphi^Y(t, y)$ and the PDEs (5.8) reduce to

$$\begin{aligned}
&\varphi^Y_t(t, y) + \varphi^Y_y(t, y)yr + \frac{1}{2}\varphi^Y_{yy}(t, y)y^2\sigma_Y^2 + A(t, y) - \varphi^Y(t, y)r = 0,\\
&\varphi(T, y) = S(y).
\end{aligned} \tag{A.20}$$

By Theorem 1 in Heath & Schweizer (2000) there exists a unique solution to (A.20) with the representation

$$\varphi^Y(t, y) = \mathbb{E}^{\hat{\mathbb{Q}}}_{t,y}\Big[\int_t^T e^{-r(s-t)}dB^Y(s)\Big].$$

By direct substitution and additivity of derivatives, we can also show that the functions $\psi^k(t, y, f) = \varphi^Y(t, y) + \varphi^k(t, y, f)$ satisfy the PDEs (5.8) for $\tilde{B}$. Consequently, $\psi^k(t, y, f) = \tilde{\varphi}^k(t, y, f)$ and the valuation operator is market-consistent.

For an orthogonal process $B^N$ we expect that $\varphi^{B^N,k}(t, y, f) = \varphi^{B^N,k}(t)$. The system of PDEs (5.8) reduces to

$$\begin{aligned}
&\varphi^k_t(t) + kA(t) + \big(\varphi^{k-1}(t) + D(t) - \varphi^k(t)\big)k\lambda(t) - \varphi^k(t)r\\
&+\Phi^k\big(t, 0, \varphi^{k-1}(t) + D(t) - \varphi^k(t)\big) = 0,\\
&\varphi^k(T) = kS.
\end{aligned} \tag{A.21}$$

Clearly, $\varphi^0(t) = 0$. If the PDEs (A.21) have solutions $(\varphi^k)_{k=1,\dots,n}$, which are derived recursively, then we can conclude that $\varphi^k$ indeed only depends on $(t, k)$ and the future claims $A, D, S$ from the benefit stream $B^N$. We now use (5.9). For the standard deviation risk margin we have

$$\begin{aligned}
\varphi^k(t) &= \mathbb{E}^{\mathbb{P}}_{t,k}\Big[\int_t^T e^{-r(s-t)}dB^N(s)\\
&\quad +\frac{1}{2}\gamma\int_t^T e^{-r(s-t)}\big|\varphi^{J(s-)-1}(s) + D(s) - \varphi^{J(s-)}(s)\big|\sqrt{J(s-)\lambda(s)}ds\Big]\\
&= \mathbb{E}^{\mathbb{P}}_{t,k}\Big[\int_t^T e^{-r(s-t)}dB^N(s)\Big] + RM^{act}_{B^N(t,T]}(t),
\end{aligned}$$

and for the variance risk margin we find

$$\begin{aligned}\varphi^k(t) &= \mathbb{E}^{\mathbb{P}}_{t,k}\Big[\int_t^T e^{-r(s-t)}dB^N(s)\\&\quad+\frac{1}{2}\gamma\int_t^T e^{-r(s-t)}\big|\varphi^{J(s-)-1}(s)+D(s)-\varphi^{J(s-)}(s)\big|^2 J(s-)\lambda(s)ds\Big]\\&= \mathbb{E}^{\mathbb{P}}_{t,k}\Big[\int_t^T e^{-r(s-t)}dB^N(s)\Big]+RM^{act}_{B^N(t,T]}(t),\end{aligned}$$

where in both cases the risk margin valuation operator $RM^{act}_{B^N(t,T]}$ is actuarial. More precisely, it takes the form (2.15) and the valuation operator is actuarial since (2.13) holds. Similar arguments can be applied when we derive a solution to the PDEs (5.8) for an orthogonal process $B^O$. □

**Proof of Theorem 5.3:** The formula for $\vartheta^*$ is deduced from (A.10) and (A.11). Using the properties of the valuation operator $\varphi$ discussed in the proof of Theorem 5.2, we can easily prove that our hedging strategy $\vartheta^*$ is market-consistent and actuarial. □