

**Asymptotic optimality of a first-order
approximate strategy for an exponential
utility maximization problem with a small
coefficient of wealth-dependent risk
aversion**

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Abstract: In Delong (2019) we investigate an exponential utility maximization problem for an insurer who faces a stream of non-hedgeable claims. We assume that the insurer's risk aversion coefficient consists of a constant risk aversion and a small amount of wealth-dependent risk aversion. We apply perturbation theory and expand the equilibrium value function of the optimization problem on the parameter ϵ controlling the degree of the insurer's risk aversion depending on wealth. We derive a candidate for the first-order approximation to the equilibrium investment strategy. In this paper we formally show that the zeroth-order investment strategy π_0^* postulated by Delong (2019) performs better than any strategy π_0 when we compare the asymptotic expansions of the objective functions up to order $\mathcal{O}(1)$ as $\epsilon \rightarrow 0$, and the first-order investment strategy $\pi_0^* + \pi_1^* \epsilon$ postulated by Delong (2019) is the equilibrium strategy in the class of strategies $\pi_0^* + \pi_1 \epsilon$ when we compare the asymptotic expansions of the objective functions up to order $\mathcal{O}(\epsilon^2)$ as $\epsilon \rightarrow 0$, where ϵ denotes the parameter controlling the degree of the insurer's risk aversion depending on wealth.

Keywords: Wealth-dependent risk aversion, PDEs, perturbation theory, asymptotic optimality.

1 Introduction

In Delong (2019) we investigate an exponential utility maximization problem for an insurer who faces a stream of non-hedgeable claims. We assume that the insurer's risk aversion coefficient changes in time and depends on the current insurer's net asset value (the excess of assets over liabilities). Since the optimization problem is time-inconsistent, we follow the game-theoretic approach developed by Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Björk and Murgoci (2014) and Björk et al. (2017). We use the notion of an equilibrium strategy and derive the HJB equation for the equilibrium value function. In order to solve the HJB equation, we use perturbation theory. We assume that the insurer's risk aversion coefficient consists of a constant risk aversion and a small amount of wealth-dependent risk aversion. The equilibrium value function is expanded on the parameter ϵ controlling the degree of the insurer's risk aversion depending on wealth. We derive candidates for the first-order approximations to the equilibrium value function and the equilibrium investment strategy.

Delong (2019) proves a lot of results which are essential to characterize the first-order approximation to the equilibrium investment strategy and justify the choice of his investment strategy as the first-order approximation. However, the order of the error of approximating the true equilibrium investment strategy with the candidate first-order approximate solution has not been proved. In this paper we formally study an asymptotic optimality of the investment strategy postulated by Delong (2019). More precisely, we show that the zeroth-order investment strategy π_0^* postulated by Delong (2019) performs better than any strategy π_0 when we compare the asymptotic expansions of the objective functions up to order $\mathcal{O}(1)$ as $\epsilon \rightarrow 0$, and the first-order investment strategy $\pi_0^* + \pi_1^* \epsilon$ postulated by Delong (2019) is the equilibrium strategy in the class of strategies $\pi_0^* + \pi_1 \epsilon$ when we compare the asymptotic expansions of the objective functions up to order $\mathcal{O}(\epsilon^2)$ as $\epsilon \rightarrow 0$, where ϵ denotes the parameter controlling the degree of the insurer's risk aversion depending on wealth. These results complete the results from Delong (2019) and give a more rigorous justification for the strategy derived in Delong (2019).

To the best of our knowledge, there are only two papers by Dong and Sircar (2014) and Delong (2019) which study exponential utility maximization problems for investors with wealth-dependent risk aversion coefficients. Moreover, the first-order approximation to the equilibrium investment strategy postulated by Delong (2019) is a new investment strategy and its properties are worth investigating.

Perturbation techniques have been popularized in financial mathematics by Fouque

et al. (2011), Fouque et al. (2014), Fouque and Hu (2017), Fouque et al. (2017). In particular, an asymptotic optimality of a candidate strategy in the class of strategies given by $\pi_0 + \pi_1\epsilon$ is investigated by Fouque and Hu (2017) in a model where an investor maximizes an expected utility in a market with stochastic volatility. The idea to study the asymptotic expansions of the objective function up to orders $\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2)$ as $\epsilon \rightarrow 0$ and an asymptotic optimality of the candidate strategy in the class of strategies given by $\pi_0 + \pi_1\epsilon$ is taken from Fouque and Hu (2017). However, the techniques which we use in this paper are different from the techniques used by Fouque and Hu (2017) since the models are different. Moreover, we deal with an equilibrium strategy, which is not the optimal strategy in the Bellman's sense, and we introduce a new asymptotic criterion for the equilibrium in order to formalize our asymptotic results.

In Sections 2-4 we briefly recall the model and the main results from Delong (2019) for reader's convenience. The results from Delong (2019) are used in the proofs in this paper. In Section 5 we prove the main result of this paper and we study the asymptotic optimality, in an appropriate sense, of the investment strategy from Delong (2019).

2 The financial and insurance model

We deal with a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and a finite time horizon $T < \infty$. On the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ we define a standard two-dimensional Brownian motion $(W, B) = (W(t), B(t), 0 \leq t \leq T)$ and a càdlàg (right-continuous with left limits) counting process $N = (N(t), 0 \leq t \leq T)$. We assume that

(A1) The filtration $\mathcal{F}_t = \bigcap_{\epsilon > 0} (\mathcal{F}_{t+\epsilon}^{W,B} \vee \mathcal{F}_{t+\epsilon}^N)$, $0 \leq t \leq T$, where $\mathcal{F}_t^{W,B} = \sigma(W(u), B(u), u \in [0, t])$, $\mathcal{F}_t^N = \sigma(N(u), u \in [0, t])$. Moreover, $\mathcal{F}_t^{W,B}$ and \mathcal{F}_t^N are independent.

The filtration \mathbb{F} is right-continuous and completed with sets of measure zero. In the sequel, the conditional expected value is denoted by $\mathbb{E}_y[\cdot] = \mathbb{E}[\cdot | Y(t) = y]$ where Y denotes the stochastic process which is used in the conditional expected value.

The financial market consists of a risk-free deposit $D = (D(t), 0 \leq t \leq T)$ and two risky indices: $S = (S(t), 0 \leq t \leq T)$, $P = (P(t), 0 \leq t \leq T)$. The value of the risk-free deposit is constant, i.e.:

$$D(t) = 1, \quad 0 \leq t \leq T. \quad (2.1)$$

The prices of the risky indices are modelled with correlated geometric Brownian mo-

tions:

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW(t), \quad 0 \leq t \leq T, \\ S(0) &= s_0,\end{aligned}\tag{2.2}$$

$$\begin{aligned}\frac{dP(t)}{P(t)} &= a dt + b \left(\rho dW(t) + \sqrt{1 - \rho^2} dB(t) \right), \quad 0 \leq t \leq T, \\ P(0) &= p_0,\end{aligned}\tag{2.3}$$

where μ, a, σ, b are positive constants which denote drifts and volatilities, and $\rho \in [-1, 1]$ denotes the correlation coefficient between the log-returns of S and P . The insurance company can invest in the deposit D and in the index S . The index P is not available for trading. The index P is the underlying investment fund for the insurance contracts sold by the insurance company, see below for a detailed description.

The insurance company keeps a homogeneous portfolio consisting of n unit-linked policies. The counting process N is used to count the number of deaths in the insurance portfolio. We assume that the lifetimes of the policyholders are independent and exponentially distributed, i.e. we assume that

$$(A2) \quad \left(N(t) - \int_0^t (n - N(s-)) \lambda ds, \quad 0 \leq t \leq T \right) \text{ is an } \mathbb{F}\text{-martingale, where } \lambda > 0.$$

Parameter λ denotes the mortality intensity in the population of the policyholders. We will use the process

$$J(t) = n - N(t), \quad 0 \leq t \leq T,$$

which counts the number of policies in force in the insurance portfolio.

The insurer faces a stream of non-hedgeable claims which is modelled with the process $C = (C(t), 0 \leq t \leq T)$ given by the equation

$$\begin{aligned}C(t) &= \int_0^t (n - N(s-)) \alpha(P(s)) ds + \int_0^t \beta(P(s)) dN(s) \\ &\quad + (n - N(T)) \eta(P(T)) \mathbf{1}_{t=T}, \quad 0 \leq t \leq T.\end{aligned}\tag{2.4}$$

Each policyholder in the insurance portfolio is entitled to three types of benefits: annuity α paid as long as the policyholder lives, life insurance benefit β paid if the policyholder dies and endowment benefit η paid if the policyholder survives till the terminal time T . The benefits α, β, η are contingent on the non-tradeable index P . We assume that

(A3) the functions $\alpha, \beta, \eta : (0, \infty) \mapsto [0, \infty)$ are bounded and Lipschitz continuous.

In order to fulfill the future liabilities, the insurer must hold a reserve. The reserve is set for the policies in force. The reserve is defined by

$$F^k(t, p) = \mathbb{E}_{t,p,k}^{\tilde{\mathbb{Q}}} [C(T) - C(t)],$$

$$(t, p, k) \in [0, T] \times (0, \infty) \times \{0, 1, \dots, n\}, \quad (2.5)$$

where $\tilde{\mathbb{Q}}$ denotes a pricing measure for C . Here, by reserve we mean an amount of money which the insurer sets aside to cover the future claims. We don't make any assumptions on the pricing measure $\tilde{\mathbb{Q}}$ in (2.5). However, we assume that

(A4) $F^k(t, p) = kF^1(t, p)$, $(t, p, k) \in [0, T] \times (0, \infty) \times \{0, \dots, n\}$, and the function $F^1 : [0, T] \times (0, \infty) \mapsto [0, \infty)$ is $\mathcal{C}^{1,2}([0, T] \times (0, \infty))$.

In the sequel, the reserve for one policy in force F^1 is simply denoted by F .

For a detailed description of the financial and insurance model and a motivation for the optimization problem we refer to Delong (2019).

3 The optimization problem and the HJB equation

Let $\pi := (\pi(t), 0 \leq t \leq T)$ denote an investment strategy which specifies the amount of money that the insurer invests in the index S . The wealth process of the insurer, denoted by $X^\pi = (X^\pi(t), 0 \leq t \leq T)$, satisfies the SDE

$$dX^\pi(t) = \pi(t) \left(\mu dt + \sigma dW(t) \right) - J(s-) \alpha(P(s)) ds + \beta(P(s)) dJ(s), \quad 0 \leq t \leq T,$$

$$X(0) = x. \quad (3.1)$$

where $x > 0$ denotes the initial wealth. The survival benefits η are subtracted from $X^\pi(T)$ at the terminal time T .

We study the time-inconsistent optimization problem:

$$\sup_{\pi} \mathbb{E} \left[-e^{-\Gamma (X^\pi(t) - J(t)F(t, P(t)))} \cdot (X^\pi(T) - J(T)\eta(P(T))) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad (3.2)$$

where Γ denotes a time-varying risk aversion coefficient which value at time t depends on the process

$$R(t) = X^\pi(t) - J(t)F(t, P(t)), \quad 0 \leq t \leq T.$$

The process R is interpreted as the insurer's net asset value - the excess of the insurer's assets over his liabilities. By the liability we mean the value of the reserve (2.5). The optimization problem (3.2) is called an exponential utility maximization problem with wealth-dependent risk aversion. We assume that the risk aversion coefficient in (3.2) satisfies the condition:

(A5) $\Gamma : \mathbb{R} \mapsto (0, \infty)$ is bounded, decreasing, Lipschitz continuous and $\mathcal{C}^2(\mathbb{R})$.

Using (2.3) and (3.1), we can deduce that the dynamics of the net asset value process R is given by the SDE

$$\begin{aligned} dR(t) = & \pi(t) \left(\mu dt + \sigma dW(t) \right) - J(t-) \alpha(P(t)) dt + \beta(P(t)) dJ(t) \\ & - J(t-) F_t(t, P(t)) dt \\ & - J(t-) F_p(t, P(t)) P(t) \left(a + b(\rho dW(t) + \sqrt{1 - \rho^2} dB(t)) \right) \\ & - J(t-) \frac{1}{2} F_{pp}(t, P(t)) b^2 P^2(t) dt - F(t, P(t)) dJ(t), \quad 0 \leq t \leq T. \end{aligned}$$

In this paper we consider a special structure of the wealth-dependent risk aversion coefficient Γ . We choose

$$\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon, \quad r \in \mathbb{R}, \quad \epsilon > 0. \quad (3.3)$$

We assume that the insurer's risk aversion coefficient Γ consists of a constant risk aversion $\gamma_0 > 0$ and a small amount $\epsilon > 0$ of wealth-dependent risk aversion γ_1 . Similar to (A5), we impose the condition:

(A6) The function $\gamma_1 : \mathbb{R} \mapsto \mathbb{R}$ is bounded, decreasing, Lipschitz continuous and $\mathcal{C}^2(\mathbb{R})$. Moreover, $\gamma_1(0) = 0$.

The assumption (3.3) allows us to apply perturbation theory and find the first-order approximation to the true solution to the optimization problem (3.2) for small $\epsilon > 0$.

Let us introduce the set of admissible investment strategies for our optimization problem (3.2).

Definition 3.1. A strategy $\pi = (\pi(t), 0 \leq t \leq T)$ is called admissible, $\pi \in \mathcal{A}$, if it satisfies the following conditions:

1. $\pi : [0, T] \times \Omega \rightarrow \mathbb{R}$ is an \mathbb{F} -predictable process determined with a measurable mapping $\pi : [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, \dots, n\} \mapsto \mathbb{R}$ such that $\pi(t) = \pi^{J(t-)}(t, X^\pi(t-), P(t))$,
2. The process $\left(\int_0^t \pi(s) dW(s), 0 \leq t \leq T \right)$ is a $BMO(\mathbb{F})$ -martingale,

3. The stochastic differential equation (3.1) has a unique solution X^π on $[0, T]$,
4. $\mathbb{E} \left[e^{-\Gamma(r)} (X^\pi(T) - J(T)\eta(P(T))) | \mathcal{F}_t \right] < \infty$, for all $t \in [0, T]$ and all $r \in \mathbb{R}$, including $\Gamma(-\infty) = \sup_{r \in \mathbb{R}} \Gamma(r)$ and $\Gamma(+\infty) = \inf_{r \in \mathbb{R}} \Gamma(r)$.

We can now define the objective function for (3.2):

$$\begin{aligned} v^{k,\pi}(t, x, p) &= \mathbb{E}_{t,x,p,k} \left[- e^{-\Gamma(x - kF(t,p))} (X^\pi(T) - J(T)\eta(P(T))) \right], \\ &\quad (t, x, p, k) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}, \quad \pi \in \mathcal{A}, \end{aligned} \quad (3.4)$$

and the auxiliary objective function:

$$\begin{aligned} w^{k,\pi}(t, x, p, r) &= \mathbb{E}_{t,x,p,k} \left[- e^{-\Gamma(r)} (X^\pi(T) - J(T)\eta(P(T))) \right], \\ &\quad (t, x, p, r, k) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R} \times \{0, 1, \dots, n\}, \quad \pi \in \mathcal{A}. \end{aligned} \quad (3.5)$$

Clearly, we have the relation:

$$v^{k,\pi}(t, x, p) = w^{k,\pi}(t, x, p, x - kF(t, p)). \quad (3.6)$$

We look for the sub-game perfect Nash equilibrium in the game with the reward given by (3.4), see e.g. Björk et al. (2017).

Definition 3.2. *Let us consider an admissible strategy $\pi^* \in \mathcal{A}$. Fix an arbitrary point $(t, x, p, k) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \{0, 1, \dots, n\}$ and choose an admissible strategy $\pi \in \mathcal{A}$. For $\delta > 0$ we define a new admissible strategy*

$$\pi^\delta(s) = \begin{cases} \pi(s), & t \leq s \leq t + \delta, \\ \pi^*(s), & t + \delta < s \leq T. \end{cases}$$

If

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(v^{k,\pi^*}(t, x, p) - v^{k,\pi^\delta}(t, x, p) \right) \geq 0, \quad (3.7)$$

for all $(t, x, p, k) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \{0, 1, \dots, n\}$ and all $\pi \in \mathcal{A}$, then π^* is called the equilibrium strategy and v^{k,π^*} is called the equilibrium value function corresponding to the equilibrium strategy π^* .

We finish with section by presenting the HJB equation and a verification theorem for the time-inconsistent optimization problem (3.2). First, we introduce operators associated with the continuous parts of the processes (X^π, P, R) .

Definition 3.3. Let \mathcal{L}_k^π and \mathcal{M}_k^π denote second order differential operators given by

$$\begin{aligned}
\mathcal{L}_k^\pi \phi(t, x, p) &= \phi_x(t, x, p)(\pi\mu - k\alpha(p)) + \frac{1}{2}\phi_{xx}(t, x, p)\pi^2\sigma^2 \\
&\quad + \phi_{px}(t, x, p)\pi bp\sigma\rho + \phi_p(t, x, p)ap + \frac{1}{2}\phi_{pp}(t, x, p)b^2p^2, \\
\mathcal{M}_k^\pi \phi(t, x, p, r) &= \mathcal{L}_k^\pi \phi(t, x, p, r) \\
&\quad + \phi_r(t, x, p, r)\left(\pi\mu - k\alpha(p) - kF_t(t, p) - kF_p(t, p)ap - \frac{1}{2}kF_{pp}(t, p)b^2p^2\right) \\
&\quad + \frac{1}{2}\phi_{rr}(t, x, p, r)\left(\pi^2\sigma^2 + (kF_p(t, p))^2b^2p^2 - 2\pi kF_p(t, p)bp\sigma\rho\right) \\
&\quad + \phi_{rp}(t, x, p, r)\left(\pi bp\sigma\rho - kF_p(t, p)b^2p^2\right) \\
&\quad + \phi_{rx}(t, x, p, r)\left(\pi^2\sigma^2 - \pi kF_p(t, p)bp\sigma\rho\right).
\end{aligned}$$

The operators \mathcal{L}_k^π and \mathcal{M}_k^π are defined, respectively, for $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$ and $\phi \in \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$. The operator $\mathcal{L}_k^\pi \phi(t, x, p, r)$ only acts on (t, x, p) and r is kept as a constant.

Theorem 3.1. [Theorem 3.1 from Delong (2019)]. Let (A1)-(A5) hold. Assume there exist functions $(v^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$, $(w^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$ and an admissible strategy $\pi^* = (\pi^{k,*})_{k=0}^n \in \mathcal{A}$ which solve the system of HJB equations:

$$\begin{aligned}
v_t^k(t, x, p) + \sup_{\pi} \left\{ \mathcal{L}_k^\pi v^k(t, x, p) - \mathcal{M}_k^\pi w^k(t, x, p, x - kF(t, p)) \right. \\
\left. + \mathcal{L}_k^\pi w^k(t, x, p, x - kF(t, p)) \right\} + \left(v^{k-1}(t, x - \beta(p), p) - v^k(t, x, p) \right) k\lambda \\
+ \left(w^{k-1}(t, x - \beta(p), p, x - kF(t, p)) \right. \\
\left. - w^{k-1}(t, x - \beta(p), p, x - \beta(p) - (k-1)F(t, p)) \right) k\lambda = 0, \\
(t, x, p) \in [0, T] \times \mathbb{R} \times (0, \infty), \\
v^k(T, x, p) = -e^{-\Gamma(x - k\eta(p))(x - k\eta(p))}, \quad (x, p) \in \mathbb{R} \times (0, \infty), \\
\pi^{k,*} = \arg \sup_{\pi} \left\{ \mathcal{L}_k^\pi v^k(t, x, p) - \mathcal{M}_k^\pi w^k(t, x, p, x - kF(t, p)) \right. \\
\left. + \mathcal{L}_k^\pi w^k(t, x, p, x - kF(t, p)) \right\}, \\
(t, x, p) \in [0, T] \times \mathbb{R} \times (0, \infty), \tag{3.8}
\end{aligned}$$

and

$$\begin{aligned}
& w_t^k(t, x, p, r) + \mathcal{L}_k^{\pi^{k,*}} w^k(t, x, p, r) \\
& + \left(w^{k-1}(t, x - \beta(p), p, r) - w^k(t, x, p, r) \right) k\lambda = 0, \\
& (t, x, p) \in [0, T) \times \mathbb{R} \times (0, \infty), \quad r \in \mathbb{R}, \\
& w^k(T, x, p, r) = -e^{-\Gamma(r)(x - k\eta(p))}, \quad (x, p) \in \mathbb{R} \times (0, \infty), \quad r \in \mathbb{R}, \tag{3.9}
\end{aligned}$$

for $k \in \{0, 1, \dots, n\}$. In addition, we assume that the families

$$\begin{aligned}
& \left\{ v^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \\
& \left\{ w^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \\
& \left\{ w^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), r), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \quad r \in \mathbb{R},
\end{aligned}$$

are uniformly integrable for any $\pi \in \mathcal{A}$. The strategy $\pi^* = (\pi^{k,*})_{k=0}^n$ is the equilibrium strategy for the time-inconsistent optimization problem (3.2) with wealth-dependent risk aversion coefficient, and $v^k(t, x, p) = v^{k,\pi^*}(t, x, p)$ is the equilibrium value function corresponding to the equilibrium strategy π^* .

Due to jumps generated by the process J , we consider a sequence of value functions and a sequence of investment strategies index with k - the number of policies in force.

4 The candidate first-order approximate strategy

Perturbation theory can be applied if our problem can be formulated by adding a small term to some parameter of a related and exactly solvable problem. Then, the solution to the main problem can be expanded in powers of this small parameter around the solution to the solvable problem. Since our risk aversion coefficient (3.3) consists of a constant risk aversion and a small amount of wealth-dependent risk aversion, we expect that the solution to the time-inconsistent exponential utility maximization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ should be expanded in powers of ϵ around the solution to the time-consistent exponential utility maximization problem with the constant risk aversion γ_0 .

We will use functions of order $\mathcal{O}(\epsilon^\theta)$. Let us recall that

$$z^\epsilon(x) \sim \mathcal{O}(\epsilon^\theta) \text{ as } \epsilon \rightarrow 0 \quad \text{if} \quad |z^\epsilon(x)| \leq K\epsilon^\theta, \quad 0 \leq \epsilon \leq \epsilon_0, \tag{4.1}$$

for some $\epsilon_0 > 0$, where K is independent of ϵ but may depend on (x, ϵ_0) .

If we assume the risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ with small $\epsilon > 0$ is used in the exponential utility function, then we can postulate the following first-order expansions for the solutions to the HJB equations (3.8)-(3.9):

$$\begin{aligned} v^k(t, x, p) &= v_0^k(t, x, p) + v_1^k(t, x, p)\epsilon + \mathcal{O}(\epsilon^2), \quad \epsilon \rightarrow 0, \\ &\quad (t, p, k) \in [0, T] \times (0, \infty) \times \{0, 1, \dots, n\}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} w^k(t, x, p, r) &= w_0^k(t, x, p) + w_1^k(t, x, p, r)\epsilon + \mathcal{O}(\epsilon^2), \quad \epsilon \rightarrow 0, \\ &\quad (t, x, p, k) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}. \end{aligned} \quad (4.3)$$

We also assume that derivatives of $(v^k)_{k=0}^n, (w^k)_{k=0}^n$ satisfy the first-order expansions of the same form (4.2)-(4.3). From equation (3.8), we can deduce the first-order expansion for the equilibrium strategy:

$$\begin{aligned} \pi^{k,*}(t, x, p) &= \pi_0^{k,*}(t, x, p) + \pi_1^{k,*}(t, x, p)\epsilon + \mathcal{O}(\epsilon^2), \quad \epsilon \rightarrow 0, \\ &\quad (t, x, p, k) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} \pi_0^{k,*}(t, x, p) &= -\frac{v_{0,x}^k(t, x, p)\mu + v_{0,px}^k(t, x, p)bp\sigma\rho}{v_{0,xx}^k(t, x, p)\sigma^2}, \quad (4.5) \\ \pi_1^{k,*}(t, x, p) &= \frac{v_{0,x}^k(t, x, p)\mu + v_{0,px}^k(t, x, p)bp\sigma\rho}{(v_{0,xx}^k(t, x, p))^2\sigma^2} \\ &\quad \times \left(v_{1,xx}^k(t, x, p) - w_{1,rr}^k(t, x, p, x - kF(t, p)) - 2w_{1,xr}^k(t, x, p, x - kF(t, p)) \right) \\ &\quad - \frac{\left(v_{1,x}^k(t, x, p) - w_{1,r}^k(t, x, p, x - kF(t, p)) \right)\mu}{v_{0,xx}^k(t, x, p)\sigma^2} \\ &\quad - \frac{\left(v_{1,px}^k(t, x, p) - w_{1,pr}^k(t, x, p, x - kF(t, p)) \right)bp\sigma\rho}{v_{0,xx}^k(t, x, p)\sigma^2} \\ &\quad - \frac{\left(w_{1,rr}^k(t, x, p, x - kF(t, p)) + w_{1,xr}^k(t, x, p, x - kF(t, p)) \right)kF_p(t, p)bp\sigma\rho}{v_{0,xx}^k(t, x, p)\sigma^2}. \end{aligned} \quad (4.6)$$

We substitute the expansions for $(v^k)_{k=0}^n, (w^k)_{k=0}^n$ and $(\pi^{k,*})_{k=0}^n$ into the system of HJB equations (3.8)-(3.9). We collect the terms of order $\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2)$ and set them to zero. We can derive the system of PDEs:

$$\begin{aligned} v_{0,t}^k(t, x, p) + \mathcal{L}_k^{\pi_0^{k,*}} v_0^k(t, x, p) + \left(v_0^{k-1}(t, x - \beta(p), p) - v_0^k(t, x, p) \right) k\lambda &= 0, \\ (t, x, p) &\in [0, T] \times \mathbb{R} \times (0, \infty), \\ v_0^k(T, x, p) &= -e^{-\gamma_0(x - k\eta(p))}, \quad (x, p) \in \mathbb{R} \times (0, \infty), \end{aligned} \quad (4.7)$$

$$\begin{aligned}
& v_{1,t}^k(t, x, p) + \mathcal{L}_k^{\pi_0^{k,*}} v_1^k(t, x, p) - \mathcal{M}_k^{\pi_0^{k,*}} w_1^k(t, x, p, x - kF(t, p)) \\
& + \mathcal{L}_k^{\pi_0^{k,*}} w_1^k(t, x, p, x - kF(t, p)) + \left(v_1^{k-1}(t, x - \beta(p), p) - v_1^k(t, x, p) \right) k\lambda \\
& + \left(w_1^{k-1}(t, x - \beta(p), p, x - kF(t, p)) \right. \\
& \quad \left. - w_1^{k-1}(t, x - \beta(p), p, x - \beta(p) - (k-1)F(t, p)) \right) k\lambda = 0, \\
& (t, x, p) \in [0, T) \times \mathbb{R} \times (0, \infty), \\
& v_1^k(T, x, p) = \gamma_1(x - k\eta(p))(x - k\eta(p))e^{-\gamma_0(x - k\eta(p))}, \quad (x, p) \in \mathbb{R} \times (0, \infty), \quad (4.8)
\end{aligned}$$

$$\begin{aligned}
& w_{0,t}^k(t, x, p) + \mathcal{L}_k^{\pi_0^{k,*}} w_0^k(t, x, p) + \left(w_0^{k-1}(t, x - \beta(p), p) - w_0^k(t, x, p) \right) k\lambda = 0, \\
& (t, x, p) \in [0, T) \times \mathbb{R} \times (0, \infty), \\
& w_0^k(T, x, p) = -e^{-\gamma_0(x - k\eta(p))}, \quad (x, p) \in \mathbb{R} \times (0, \infty), \quad (4.9)
\end{aligned}$$

$$\begin{aligned}
& w_{1,t}^k(t, x, p, r) + \mathcal{L}_k^{\pi_0^{k,*}} w_1^k(t, x, p, r) + \left(w_1^{k-1}(t, x - \beta(p), p, r) - w_1^k(t, x, p, r) \right) k\lambda = 0 \\
& (t, x, p) \in [0, T) \times \mathbb{R} \times (0, \infty), \quad r \in \mathbb{R}, \\
& w_1^k(T, x, p, r) = \gamma_1(r)(x - k\eta(p))e^{-\gamma_0(x - k\eta(p))}, \quad (x, p) \in \mathbb{R} \times (0, \infty), \quad r \in \mathbb{R}, \quad (4.10)
\end{aligned}$$

for $k = 0, 1, \dots, n$. We can find the solutions to the PDEs (4.7)-(4.10). These solutions are given by

$$v_0^k(t, x, p) = -e^{-\gamma_0 x} e^{\gamma_0 h^{k, \gamma_0}(t, p)}, \quad (4.11)$$

$$w_0^k(t, x, p) = -e^{-\gamma_0 x} e^{\gamma_0 h^{k, \gamma_0}(t, p)}, \quad (4.12)$$

$$\begin{aligned}
v_1^k(t, x, p) &= \gamma_1(x - kF(t, p)) \\
&\quad \times \left(x - h^{k, \gamma_0}(t, p) - \gamma_0 g^{k, \gamma_0}(t, p) \right) e^{-\gamma_0 x} e^{\gamma_0 h^{k, \gamma_0}(t, p)}, \quad (4.13)
\end{aligned}$$

$$w_1^k(t, x, p, r) = \gamma_1(r) \left(x - h^{k, \gamma_0}(t, p) - \gamma_0 g^{k, \gamma_0}(t, p) \right) e^{-\gamma_0 x} e^{\gamma_0 h^{k, \gamma_0}(t, p)}, \quad (4.14)$$

where the functions $(h^k)_{k=0}^n$ solve the PDEs:

$$\begin{aligned}
& h_t^k(t, p) + \left(a - \frac{\mu b \rho}{\sigma} \right) p h_p^k(t, p) + \frac{1}{2} b^2 p^2 h_{pp}^k(t, p) + k\alpha(p) - \frac{\mu^2}{2\sigma^2 \gamma} - \frac{k\lambda}{\gamma} \\
& + \frac{e^{\gamma \beta(p)} e^{\gamma h^{k-1}(t, p)}}{\gamma} k\lambda e^{-\gamma h^k(t, p)} \\
& + \frac{1}{2} \gamma (1 - \rho^2) b^2 p^2 (h_p^k(t, p))^2 = 0, \quad (t, p) \in [0, T) \times (0, \infty), \\
& h^k(T, p) = k\eta(p), \quad p \in (0, \infty), \quad k \in \{0, \dots, n\}, \quad (4.15)
\end{aligned}$$

and the functions $(g^k)_{k=0}^n$ solve the PDEs:

$$\begin{aligned}
& g_t^k(t, p) + \left(a - \frac{\mu b \rho}{\sigma} + \gamma(1 - \rho^2) b^2 p h_p^k(t, p) \right) p g_p^k(t, p) + \frac{1}{2} b^2 p^2 g_{pp}^k(t, p) \\
& - e^{\gamma(\beta(p) + h^{k-1}(t, p) - h^k(t, p))} k \lambda g^k(t, p) \\
& + \frac{\mu^2}{2\sigma^2 \gamma^2} + \frac{1}{2} (1 - \rho^2) b^2 p^2 (h_p^k(t, p))^2 \\
& + \frac{e^{\gamma(\beta(p) + h^{k-1}(t, p) - h^k(t, p))} \left(\gamma \left(\beta(p) + h^{k-1}(t, p) - h^k(t, p) \right) - 1 \right) + 1}{\gamma^2} k \lambda \\
& + e^{\gamma(\beta(p) + h^{k-1}(t, p) - h^k(t, p))} k \lambda g^{k-1}(t, p) = 0, \quad (t, p) \in [0, T) \times (0, \infty), \\
& g^k(T, p) = 0, \quad p \in (0, \infty), \quad k \in \{0, \dots, n\}.
\end{aligned} \tag{4.16}$$

We remark that $h^{k, \gamma_0}, g^{k, \gamma_0}$ in (4.11)-(4.14) denote the solutions to the PDEs (4.15)-(4.16) with $\gamma = \gamma_0$. Consequently, the first-order approximation to the equilibrium strategy (4.4) is determined with

$$\pi_0^{k,*}(t, p) = \frac{\mu}{\sigma^2 \gamma_0} + \frac{h_p^{k, \gamma_0}(t, p) b p \rho}{\sigma}, \tag{4.17}$$

$$\pi_1^{k,*}(t, x, p) = -\frac{\mu \gamma_1(x - kF(t, p))}{\sigma^2 \gamma_0^2} + \frac{g_p^{k, \gamma_0}(t, p) \gamma_1(x - kF(t, p)) b p \rho}{\sigma}. \tag{4.18}$$

Let us summarize the results.

Proposition 4.1. *[Propositions 5.1-5.4 from Delong (2019)] Let (A1)-(A3) hold.*

1. *There exist unique solutions $(h^k)_{k=0}^n \in \mathcal{C}([0, T] \times (0, \infty)) \cap \mathcal{C}^{1,2}([0, T] \times (0, \infty))$ to the system of PDEs (4.15). Moreover, the functions $(h^k)_{k=0}^n : [0, T] \times (0, \infty) \mapsto \mathbb{R}$ are uniformly bounded in (t, p) , and Lipschitz continuous in p uniformly in t .*

2. *In addition, assume that*

(A7) *There exist mixed derivatives $(h_{tp}^k)_{k=0}^n \in \mathcal{C}([0, T] \times (0, \infty))$.*

There exist unique solutions $(g^k)_{k=0}^n \in \mathcal{C}([0, T] \times (0, \infty)) \cap \mathcal{C}^{1,2}([0, T] \times (0, \infty))$ to the system of PDEs (4.16). Moreover, the functions $(g^k)_{k=0}^n : [0, T] \times (0, \infty) \mapsto \mathbb{R}$ are uniformly bounded in (t, p) , and Lipschitz continuous in p uniformly in t .

3. *Let us define*

$$Z(t) = \sum_{k=0}^n h_p^k(t, P(t)) P(t) \mathbf{1}\{J(t-) = k\}, \quad 0 \leq t \leq T,$$

$$\mathcal{Z}(t) = \sum_{k=0}^n g_p^k(t, P(t)) P(t) \mathbf{1}\{J(t-) = k\}, \quad 0 \leq t \leq T.$$

The processes $(\int_0^t Z(s)dW(s), 0 \leq t \leq T)$, $(\int_0^t \mathcal{Z}(s)dW(s), 0 \leq t \leq T)$ are BMO-martingales.

4. There exist solutions $(v_0^k, v_1^k, w_0^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$ and $(w_1^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$ to the PDEs (4.7)-(4.10) given by (4.11)-(4.14).

Theorem 4.1. [Theorem 6.1 from Delong (2019)] Let (A1)-(A7) hold. For a sufficiently small $\epsilon > 0$, the investment strategy given by

$$\hat{\pi}^{k,*}(t, x, p) = \pi_0^{k,*}(t, p) + \pi_1^{k,*}(t, x, p)\epsilon, \quad (4.19)$$

with (4.17)-(4.18) is admissible, i.e. $\hat{\pi}^* = (\hat{\pi}^{k,*})_{k=0}^n \in \mathcal{A}$. The investment strategy (4.19) is a candidate asymptotic first-order approximation to the equilibrium investment strategy for the optimization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ as $\epsilon \rightarrow 0$.

5 Asymptotic optimality of the candidate first-order approximate strategy

First, we specify the class of investment strategies in which we show that the investment strategy (4.19) is asymptotically optimal for our optimization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ as $\epsilon \rightarrow 0$. Next, we formalize and explain what we mean by the asymptotic optimality of (4.19) in our optimization problem. Finally, we present and prove the key result of this paper.

Definition 5.1. Let us consider the utility maximization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ with $\epsilon > 0$. A strategy $\pi := (\pi(t), 0 \leq t \leq T)$ is in the class \mathcal{B} if

1. $\pi : [0, T] \times \Omega \rightarrow \mathbb{R}$ is an \mathbb{F} -predictable process determined with a measurable mapping $\pi : [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, \dots, n\} \mapsto \mathbb{R}$ such that $\pi(t) = \pi^{J(t-)}(t, X^\pi(t-), P(t))$ and π has the representation:

$$\begin{aligned} \pi^k(t, x, p) &= \pi_0^k(t, x, p) + \pi_1^k(t, x, p)\epsilon, \\ (t, x, p, k) &\in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}, \end{aligned} \quad (5.1)$$

2. The mappings $x \mapsto \pi_i^k(t, x, P(t, \omega))$ satisfy the Lipschitz conditions:

$$\begin{aligned} & |\pi_i^k(t, x, P(t, \omega)) - \pi_i^k(t, x', P(t, \omega))| \\ & \leq H(t, \omega) |x - x'| \leq K(1 + P(t, \omega)) |x - x'|, \quad i = 0, 1, \\ & (t, x, \omega, k), (t, x', \omega, k) \in [0, T] \times \mathbb{R} \times \Omega \times \{0, 1, \dots, n\}, \end{aligned}$$

where $H : [0, T] \times \Omega \mapsto [0, \infty)$ is a continuous process, adapted to the filtration $\sigma(P(u), u \in [0, T])$, such that $(\int_0^t H(s) dW(s), 0 \leq t \leq T)$ is a BMO-martingale,

3. The mappings $x \mapsto \pi_i^k(t, x, P(t, \omega))$ satisfy the growth conditions:

$$\begin{aligned} & |\pi_i^k(t, x, P(t, \omega))| \\ & \leq H(t, \omega) \leq K(1 + P(t, \omega)), \quad i = 0, 1, \\ & (t, x, \omega, k) \in [0, T] \times \mathbb{R} \times \Omega \times \{0, 1, \dots, n\}, \end{aligned}$$

where $H : [0, T] \times \Omega \mapsto [0, \infty)$ is a continuous process, adapted to the filtration $\sigma(P(u), u \in [0, T])$, such that $(\int_0^t H(s) dW(s), 0 \leq t \leq T)$ is a BMO-martingale,

4. $\mathbb{E} \left[e^{-\Gamma(r)} (X^{\pi_0(T)} - J(T)\eta(P(T))) \middle| \mathcal{F}_t \right] < \infty$, for all $t \in [0, T]$ and all $r \in \mathbb{R}$, including $\Gamma(-\infty) = \sup_{r \in \mathbb{R}} \Gamma(r)$ and $\Gamma(+\infty) = \inf_{r \in \mathbb{R}} \Gamma(r)$.

We remark that the amount of π_1 added to π_0 , in order to define the admissible strategy (5.1), is controlled with the parameter ϵ which represents the degree of the insurer's risk aversion depending on wealth. If we choose $\pi_1 = 0$, then we can consider strategies independent of the parameter ϵ within the class \mathcal{B} . Finally, the process H in the Lipschitz and growth conditions may depend on the strategies π_0, π_1 .

Since we use perturbation techniques, the idea of which is to expand the true solution in powers of the small parameter ϵ , it is natural to consider the investment strategies of the form (5.1) in point 1 of Definition 5.1, see also Fouque and Hu (2017). Points 2-4 from Definition 5.1 are closely related to points 2-4 from Definition 3.1. Points 2-3 from Definition 5.1 describe in more details the measurable mapping $(t, x, p, k) \mapsto \pi^k(t, x, p)$ which characterizes the investment strategy. In particular, points 2-3 from Definition 5.1 imply that points 2-3 from Definition 3.1 are satisfied. They are rather standard in the theory of stochastic differential equations and backward stochastic differential equations with BMO-martingales, see Chapter V.3 in Protter (2005) and Ankirchner et al. (2007). Finally, since we add a small amount ϵ of π_1 to π_0 in order to define the strategy $\pi \in \mathcal{B}$ in (5.1), we expect that point 4 from Definition 3.1 should only be needed for π_0 (which is point 4 from Definition 5.1). In Proposition 5.1 below, we show

that $\mathcal{B} \subset \mathcal{A}$ and the candidate first-order approximation to the equilibrium strategy $\hat{\pi}^* \in \mathcal{B}$ for a sufficiently small $\epsilon > 0$. Although Definition 5.1 may look technical, we believe that it describes a very reasonable class of investment strategies which are important for our exponential utility maximization problem (3.2) with a small amount ϵ of wealth-dependent risk aversion and does not exclude any relevant strategies.

We now present the main theorem of this paper.

Theorem 5.1. *We assume that (A1)-(A7) hold. The strategies $\pi_0^* = (\pi_0^{k,*})_{k=0}^n, \pi_1^* = (\pi_1^{k,*})_{k=0}^n$ are given by (4.17)-(4.18), and the functions $(v_0^k)_{k=0}^n, (v_1^k)_{k=0}^n$ are given by (4.11), (4.13). Let $(v_\epsilon^{k,\pi})_{k=0}^n$ and $(w_\epsilon^{k,\pi})_{k=0}^n$ denote the objective functions (3.4)-(3.5) for the utility maximization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ with $\epsilon > 0$ when the strategy π is applied. We allow for strategies $\pi = (\pi^k)_{k=0}^n \in \mathcal{B}$ such that $(v_\epsilon^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$ and $(w_\epsilon^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$. We fix $(t, x, p, k) \in [0, T) \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}$.*

(i) *For any strategy π_0 , we have the asymptotic zeroth-order approximation to the objective function:*

$$v_\epsilon^{k,\pi_0}(t, x, p) = V^{k,\pi_0}(t, x, p) + \mathcal{O}(\epsilon), \quad \epsilon \rightarrow 0, \quad (5.2)$$

where $V^{k,\pi}$ denotes the objective function for the time-consistent optimization problem (3.2) with $\Gamma(r) = \gamma_0$.

(ii) *The strategy π_0^* performs better than any strategy π_0 when we compare the asymptotic approximations to the objective functions up to order $\mathcal{O}(1)$, i.e.*

$$\lim_{\epsilon \rightarrow 0} (v_\epsilon^{k,\pi_0^*}(t, x, p) - v_\epsilon^{k,\pi_0}(t, x, p)) \geq 0. \quad (5.3)$$

(iii) *For any strategy $\pi_0^* + \pi_1\epsilon$, we have the asymptotic first-order approximation to the objective function:*

$$v_\epsilon^{k,\pi_0^* + \pi_1\epsilon}(t, x, p) = v_0^k(t, x, p) + v_1^k(t, x, p)\epsilon + \mathcal{O}(\epsilon^2), \quad \epsilon \rightarrow 0, \quad (5.4)$$

(iv) *The strategy $\pi_0^* + \pi_1^*\epsilon$ is the equilibrium strategy in the class of strategies $\pi_0^* + \pi_1\epsilon$ when we compare the asymptotic approximations to the objective functions up to order $\mathcal{O}(\epsilon^2)$, i.e.*

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(\lim_{\epsilon \rightarrow 0} \frac{v_\epsilon^{k,\pi_0^* + \pi_1^*\epsilon}(t, x, p) - v_\epsilon^{k,\pi_0^* + \pi_1\epsilon}(t, x, p)}{\epsilon^2} \right) \geq 0, \quad (5.5)$$

where, for $\delta \in [0, T - t]$, we define

$$\pi_1^\delta(s) = \begin{cases} \pi_1(s), & t \leq s \leq t + \delta, \\ \pi_1^*(s), & t + \delta < s \leq T. \end{cases} \quad (5.6)$$

Remark 5.1: a) The function v_ϵ^{k, π_0} depends on ϵ since we use $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$. The function $v_\epsilon^{k, \pi_0 + \pi_1\epsilon}$ depends on ϵ since we use $\pi = \pi_0 + \pi_1\epsilon$ and $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$. The subscript ϵ in $(v_\epsilon^k)_{k=0}^n$ will be omitted in the sequel.

b) If we use $\Gamma(r) = \gamma_0$, then π_0^* is the optimal investment strategy for the time-consistent exponential utility maximization problem (3.2) with the constant risk aversion coefficient γ_0 , and the functions $(v_0^k)_{k=0}^n$ define the corresponding optimal value function, see Proposition 5.1 in Delong (2019). We note that $(v_0^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$ by Proposition 4.1.

c) We consider a class of strategies which is potentially smaller than the class \mathcal{B} since we require that the objective functions (3.4)-(3.5) are smooth for the strategies considered in Theorem 5.1. This assumption is reasonable since in this paper we work with smooth (classical) solutions to HJB equations and PDEs. In Theorem 3.1 we assume that the equilibrium value function (i.e. the objective function for our optimization problem for the equilibrium strategy) is a smooth solution to HJB equations. In Proposition 4.1 we prove that the candidate first-order approximation to the equilibrium value function is a smooth solution to PDEs. Finally, Remark b) shows that the optimal value function for the time-consistent optimization problem with constant risk aversion (i.e. the objective function for our optimization problem with $\epsilon = 0$) is also smooth.

d) Since (3.6) holds, then $w^{k, \pi} \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$ implies that $v^{k, \pi} \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$. \square

Theorem 5.1 gives a more rigorous justification for the investment strategy derived in Delong (2019). The assertions (i)-(ii) from Theorem 5.1 are intuitively clear in the view of Remark 5.1.b. The zeroth-order investment strategy π_0^* postulated in Theorem 4.1 and by Delong (2019) performs better than any strategy π_0 when we compare the asymptotic expansions of the objective functions up to order $\mathcal{O}(1)$ as $\epsilon \rightarrow 0$. If we want to study investment strategies which are series expansions in powers of ϵ , then, by perturbation theory and Remark 5.1.b., it is natural to consider expansions around the strategy π_0^* . The most interesting are the assertions (iii)-(iv) from Theorem 5.1 where we show that the first-order investment strategy $\pi_0^* + \pi_1^*\epsilon$ postulated in Theorem 4.1 and by Delong (2019) is the equilibrium strategy in a reasonable class of strategies $\pi_0^* + \pi_1\epsilon$ when we compare the asymptotic approximations to the objective functions

up to order $\mathcal{O}(\epsilon^2)$ as $\epsilon \rightarrow 0$. The criterion (5.5) is a modification of the well-established criterion (3.7) for the equilibrium in continuous-time models. In (3.7) we compare the objective functions for the exponential utility maximization problem with the risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ for the strategies π^* and π^δ . In (5.5) we use the asymptotic expansions (5.4) of the objective functions for the exponential utility maximization problem with the risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ for the strategies $\pi_0^* + \pi_1^*\epsilon$ and $\pi_0^* + \pi_1^\delta\epsilon$ and compare the terms in these expansions up to order $\mathcal{O}(\epsilon^2)$. To the best of our knowledge the criterion (5.5) is new and has not been investigated in the literature. We point out that (5.5) is not related to ϵ -equilibrium.

We now prove Theorem 5.1 by using series of lemmas and propositions.

Proposition 5.1. *Let us consider the utility maximization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ with a sufficiently small $\epsilon > 0$.*

(i) *Any strategy $\pi = \pi_0 + \pi_1\epsilon \in \mathcal{B}$ is in the class \mathcal{A} .*

(ii) *The strategies π_0^* and $\hat{\pi}^* = \pi_0^* + \pi_1^*\epsilon$ are in the class \mathcal{B} .*

Proof: Assertion (i): We choose $\pi = \pi_0 + \pi_1\epsilon \in \mathcal{B}$ from Definition 5.1. We will show that all points from Definition 3.1 are satisfied. Point 1 is obvious. Point 2 follows from the growth conditions for π_0 and π_1 . Point 3 can be deduced from Theorem V.7 in Protter (2005) since π_0 and π_1 are process Lipschitz. We are left with point 4. Let us introduce the process

$$\begin{aligned}
Y(t) = & J(T)\eta(P(T)) - \int_t^T \left(\frac{\mu^2}{2\sigma^2\gamma} - J(s-)\alpha(P(s)) + \frac{\mu}{\sigma}Z_1(s) - \frac{1}{2}\gamma(Z_2(s))^2 \right. \\
& \left. - \frac{e^{\gamma(\beta(P(s))+Q(s))} - 1}{\gamma}J(s-)\lambda \right) ds \\
& - \int_t^T Z_1(s)dW(s) - \int_t^T Z_2(s)dB(s) - \int_t^T Q(s)dN(s), \quad 0 \leq t \leq T. \quad (5.7)
\end{aligned}$$

The process Y is used to define the solution to the exponential utility maximization problem (3.2) with the constant risk aversion coefficient $\Gamma(r) = \gamma_0 = \gamma$, see Theorem 5.1 in Delong (2019). We can show that

$$v_0^k(t, x, p) = -e^{-\gamma x} e^{\gamma h^k(t, p)} = -e^{-\gamma x} e^{\gamma Y(t)}|_{P(t)=p, J(t)=k}, \quad (5.8)$$

where $v_0^k(t, x, p)$ is the optimal value function for the time-consistent exponential utility maximization problem for the initial point (t, x, p, k) .

We choose $r \in \mathbb{R}$ and set $\gamma_1 := \gamma_1(r)$. We choose $t \in [0, T]$. We have the following decomposition:

$$\begin{aligned}
& (\gamma_0 + \gamma_1 \epsilon)(X^\pi(T) - J(T)\eta(P(T))) \\
&= \gamma_0 \left(X^\pi(t) + \int_t^T \pi_0(s) \mu ds + \int_t^T \pi_0(s) \sigma dW(s) \right. \\
&\quad \left. - \int_t^T J(s-) \alpha(P(s)) ds + \int_t^T \beta(P(s)) dJ(s) - Y(T) \right) \\
&\quad + \gamma_0 \left(\int_t^T \pi_1(s) \mu ds + \int_t^T \pi_1(s) \sigma dW(s) \right) \epsilon \\
&\quad + \gamma_1 \epsilon \left(X^\pi(t) + \int_t^T (\pi_0(s) + \pi_1(s)) \mu ds \right. \\
&\quad \left. + \int_t^T (\pi_0(s) + \pi_1(s)) \sigma dW(s) \right. \\
&\quad \left. - \int_t^T J(s-) \alpha(P(s)) ds + \int_t^T \beta(P(s)) dJ(s) - J(T)\eta(P(T)) \right) \\
&= (\gamma_0 + \gamma_1 \epsilon) X^\pi(t) - \gamma_0 Y(t) \\
&\quad + \gamma_0 \left(\int_t^T \pi_0(s) \mu ds + \int_t^T \pi_0(s) \sigma dW(s) \right. \\
&\quad \left. - \int_t^T J(s-) \alpha(P(s)) ds + \int_t^T \beta(P(s)) dJ(s) - (Y(T) - Y(t)) \right) \\
&\quad + \epsilon \left(\int_t^T \tilde{\pi}(s) \mu ds + \int_t^T \tilde{\pi}(s) \sigma dW(s) \right) \\
&\quad - \gamma_1 \epsilon \left(\int_t^T J(s-) \alpha(P(s)) ds - \int_t^T \beta(P(s)) dJ(s) + J(T)\eta(P(T)) \right), \quad (5.9)
\end{aligned}$$

where we introduce the strategy

$$\tilde{\pi}(s) = \gamma_1 \pi_0(s) + (\gamma_0 + \gamma_1 \epsilon) \pi_1(s), \quad 0 \leq s \leq T.$$

From point 3 from Definition 5.1 and (A6), we deduce that the process $(\int_0^t \tilde{\pi}(s) dW(s), 0 \leq t \leq T)$ is a *BMO*-martingale, and

$$\begin{aligned}
& \left\| \int_0^T \tilde{\pi}(s) dW(s) \right\|_{BMO}^2 \\
& \leq K \left(\left\| \int_0^T \pi_0(s) dW(s) \right\|_{BMO}^2 + \left\| \int_0^T \pi_1(s) dW(s) \right\|_{BMO}^2 \right) < \infty. \quad (5.10)
\end{aligned}$$

We now study the expected value:

$$\begin{aligned}
& \mathbb{E} \left[e^{-\Gamma(r) \left(X^\pi(T) - J(T)\eta(P(T)) \right)} \middle| \mathcal{F}_t \right] \\
&= e^{-(\gamma_0 + \gamma_1 \epsilon) X^\pi(t) + \gamma_0 Y(t)} \mathbb{E} \left[e^{-\gamma_0 \left(X^{\pi_0}(T) - X^{\pi_0}(t) - (Y(T) - Y(t)) \right)} \right. \\
&\quad \times e^{-\epsilon \left(\int_t^T \tilde{\pi}(s) \mu ds + \int_t^T \tilde{\pi}(s) \sigma dW(s) \right)} \\
&\quad \left. \times e^{\gamma_1 \epsilon \left(\int_t^T J(s-) \alpha(P(s)) ds - \int_t^T \beta(P(s)) dJ(s) + J(T)\eta(P(T)) \right)} \middle| \mathcal{F}_t \right]. \tag{5.11}
\end{aligned}$$

By Hölder's inequality and boundedness of α, β, η , we can derive

$$\begin{aligned}
& \mathbb{E} \left[e^{-\Gamma(r) \left(X^\pi(T) - J(T)\eta(P(T)) \right)} \middle| \mathcal{F}_t \right] \\
&\leq K e^{-(\gamma_0 + \gamma_1 \epsilon) X^\pi(t) + \gamma_0 Y(t)} \left(\mathbb{E} \left[e^{-\gamma_0 q_1 \left(X^{\pi_0}(T) - X^{\pi_0}(t) - (Y(T) - Y(t)) \right)} \middle| \mathcal{F}_t \right] \right)^{\frac{1}{q_1}} \\
&\quad \times \left(\mathbb{E} \left[e^{-q_1^* \epsilon \left(\int_t^T \tilde{\pi}(s) \mu ds + \int_t^T \tilde{\pi}(s) \sigma dW(s) \right)} \middle| \mathcal{F}_t \right] \right)^{\frac{1}{q_1^*}}, \tag{5.12}
\end{aligned}$$

for a sufficiently small $q_1 > 1$ and its conjugate $q_1^* > 1$. We can choose a sufficiently small $q_1 > 1$ such that $\gamma_0 q_1 = \gamma_0 + \gamma_1(r)\epsilon = \Gamma(r)$, if a sufficiently small $\epsilon > 0$ is used. Consequently, by point 4 from Definition 5.1, the first expected value in (5.12) is finite. As far as the second expected value is concerned, we introduce the process

$$M(s) = e^{-\int_t^s q_1^* \epsilon \tilde{\pi}(u) \sigma dW(u) - \frac{1}{2} \int_t^s |q_1^* \epsilon \tilde{\pi}(u) \sigma|^2 du}, \quad t \leq s \leq T.$$

The process M is an exponential martingale generated by a *BMO*-martingale since (5.10) holds. Consequently, applying Hölder inequality and reverse Hölder inequality, see Theorem 3.1 in Kazamaki (1997), we get

$$\begin{aligned}
& \mathbb{E} \left[e^{-q_1^* \epsilon \left(\int_t^T \tilde{\pi}(s) \mu ds + \int_t^T \tilde{\pi}(s) \sigma dW(s) \right)} \middle| \mathcal{F}_t \right] \\
&\leq \left(\mathbb{E} \left[|M(T)|^{q_2} \middle| \mathcal{F}_t \right] \right)^{\frac{1}{q_2}} \\
&\quad \times \left(\mathbb{E} \left[e^{\frac{1}{2} \int_t^T q_2^* |q_1^* \epsilon \tilde{\pi}(s) \sigma|^2 ds - \int_t^T q_2^* q_1^* \epsilon \tilde{\pi}(s) \mu ds} \middle| \mathcal{F}_t \right] \right)^{\frac{1}{q_2}} \\
&\leq K \left(\mathbb{E} \left[e^{\frac{1}{2} \int_t^T q_2^* |q_1^* \epsilon \tilde{\pi}(s) \sigma|^2 ds - \int_t^T q_2^* q_1^* \epsilon \tilde{\pi}(s) \mu ds} \middle| \mathcal{F}_t \right] \right)^{\frac{1}{q_2^*}}, \tag{5.13}
\end{aligned}$$

for a sufficiently small $q_2 > 1$ and its conjugate $q_2^* > 1$. Finally, for a sufficiently small ϵ , we have the inequality

$$\begin{aligned}
& \mathbb{E} \left[e^{\frac{1}{2} \int_t^T q_2^* |q_1^* \epsilon \tilde{\pi}(s) \sigma|^2 ds - \int_t^T q_2^* q_1^* \epsilon \tilde{\pi}(s) \mu ds} \middle| \mathcal{F}_t \right] \\
&\leq K_1 \mathbb{E} \left[e^{K_2 \epsilon^2 \int_t^T |\tilde{\pi}(s)|^2 ds} \middle| \mathcal{F}_t \right] \leq \frac{K_1}{1 - K_2 \epsilon^2 \left\| \int_0^T \tilde{\pi}(s) dW(s) \right\|_{BMO}^2} < \infty, \tag{5.14}
\end{aligned}$$

by (5.10) and John-Nirenberg inequality, see Theorem 2.2 in Kazamaki (1997). Collecting (5.12) and (5.14), we can conclude that the expected value (5.11) is *a.s.* finite and our strategy π satisfies point 4 from Definition 3.1. Hence, $\pi \in \mathcal{B}$ implies that $\pi \in \mathcal{A}$. Assertion (ii): Point 1 from Definition 5.1 is obvious. Points 2-3 can be deduced from (A6) and the properties specified in Proposition 4.1. In particular, the properties that the mapping $p \mapsto h^k(t, p)$ is Lipschitz continuous on $(0, \infty)$ uniformly in $t \in [0, T]$ and $h^k \in \mathcal{C}([0, T] \times (0, \infty)) \cap \mathcal{C}^{1,2}([0, T] \times (0, \infty))$ imply that the derivative $(t, p) \mapsto h_p^k(t, p)$ is uniformly bounded and jointly continuous on $[0, T] \times (0, \infty)$. In the definition of the investment strategy (4.19) we can choose the left limit $\lim_{t \rightarrow T^-} h_p^k(t, P(t, \omega))$ and we have a continuous, finite mapping $t \mapsto h_p^k(t, P(t, \omega))P(t, \omega)$ on $[0, T]$ for *a.a.* ω . The same arguments hold for $g_p^k(t, p)$. We have to prove point 4. In fact, we only have to prove that the first expected value in (5.12) is finite if π_0^* is used. By Remark 5.1.b. the strategy π_0^* is the optimal investment strategy for the optimization problem (3.2) with constant risk aversion, see also Theorems 5.1, 6.1 in Delong (2019). From properties of the optimal value function (5.8) for the time-consistent exponential utility maximization problem (3.2) with the constant risk aversion $\Gamma(r) = \gamma_0$, we can deduce that

$$\mathcal{M}(s) = e^{-\gamma_0(X^{\pi_0^*}(s) - X^{\pi_0^*}(t) - (Y(s) - Y(t)))}, \quad t \leq s \leq T,$$

is an exponential martingale generated by a *BMO*-martingale, see (8.9) in Delong (2019) or a general theory in Hu et al. (2005). Hence, by reverse Hölder inequality, we can choose a sufficiently small $q_1 > 1$ such that

$$\mathbb{E}[|\mathcal{M}(T)|^{q_1} | \mathcal{F}_t] \leq K.$$

We can now use the same arguments as in the first part of the proof. □

Lemma 5.1. *Let $\pi \in \mathcal{A}$ denote an admissible strategy for the utility maximization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ with $\epsilon > 0$, and let $(v_0^k, v_1^k, w_0^k, w_1^k)_{k=0}^n$ denote the solutions to the PDEs (4.7)-(4.10). The families*

$$\begin{aligned} & \left\{ v_i^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \\ & \left\{ w_i^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \\ & \left\{ w_i^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), r), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \quad r \in \mathbb{R}, \end{aligned}$$

are uniformly integrable, for $i = 0, 1$.

Proof: The solutions to (4.7)-(4.10) are given by (4.11)-(4.14). By Proposition 4.1, the functions $(h^k)_{k=0}^n, (g^k)_{k=0}^n$ are bounded in (t, p, k) . Since γ_1 is bounded by (A6), it is sufficient to prove that $\{e^{-\gamma_0 X^\pi(\mathcal{T})}, \mathcal{T} \text{ is an } \mathbb{F}\text{-stopping time}\}$ and $\{e^{-\gamma_0 X^\pi(\mathcal{T})} X^\pi(\mathcal{T}), \mathcal{T} \text{ is an } \mathbb{F}\text{-stopping time}\}$ are uniformly integrable for any $\pi \in \mathcal{A}$.

We choose $\pi \in \mathcal{A}$. Points 2 and 4 from Definition 3.1 and the assumption (A6) that $\gamma_1(0) = 0$ imply that the family $\left\{e^{-\gamma_0 X^\pi(\mathcal{T})}, \mathcal{T} \text{ is an } \mathbb{F}\text{-stopping time}\right\}$ is uniformly integrable, see Remark 8 in Hu et al. (2005). We now consider the second family. We choose a sufficiently small $q > 1$. We have the inequality

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} \left[\left| e^{-\gamma_0 X^\pi(t)} X^\pi(t) \right|^q \right] \\ & \leq \left(\sup_{t \in [0, T]} \mathbb{E} \left[e^{-\gamma_0 \kappa q X^\pi(t)} \right] \right)^{\frac{1}{\kappa}} \times \left(\sup_{t \in [0, T]} \mathbb{E} \left[\left| X^\pi(t) \right|^{\kappa^* q} \right] \right)^{\frac{1}{\kappa^*}}, \end{aligned} \quad (5.15)$$

where we choose a sufficiently small $\kappa > 1$, and κ^* denotes its conjugate. Since we can set $\gamma_0 \kappa q = \gamma_0 + \gamma_1(r)\epsilon = \Gamma(r)$ for some $r \in \mathbb{R}$ and sufficiently small $\epsilon > 0, \kappa > 1, q > 1$, the first term in (5.15) is finite by uniform integrability of $\{e^{-\gamma_0 \kappa q X^\pi(\mathcal{T})}, \mathcal{T} \text{ is an } \mathbb{F}\text{-stopping time}\}$ (by points 2 and 4 from Definition 3.1 and the arguments from above). As far as the second term in (5.15) is concerned, let us recall the dynamics (3.1) for the process X^π . For any $\kappa^* > 1$ and $q > 1$, we have the inequalities

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} \left[\left| X^\pi(t) \right|^{\kappa^* q} \right] & \leq K \left(1 + \mathbb{E} \left[\left| \int_0^T |\pi(s)|^2 ds \right|^{\frac{\kappa^* q}{2}} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \pi(s) dW(s) \right|^{\kappa^* q} \right] \right) \\ & \leq K \left(1 + \mathbb{E} \left[\left| \int_0^T |\pi(s)|^2 ds \right|^{\frac{\kappa^* q}{2}} \right] \right) \\ & \leq K \left(1 + \left\| \int_0^T \pi(s) dW(s) \right\|_{BMO}^{2 \left[\frac{\kappa^* q}{2} \right] + 2} \right) < \infty, \end{aligned} \quad (5.16)$$

where we use the Burkholder-Davis-Gundy inequality and the energy inequality (see e.g. page 29 in Kazamaki (1997)). \square

Lemma 5.2. *Let $\pi \in \mathcal{A}$ denote an admissible strategy and $(v^{k, \pi}, w^{k, \pi})_{k=0}^n$ denote the corresponding objective functions (3.4)-(3.5) for the utility maximization problem (3.2) with the wealth-dependent risk aversion coefficient $\Gamma(r) = \gamma_0 + \gamma_1(r)\epsilon$ with $\epsilon > 0$. The*

families

$$\begin{aligned} & \left\{ v^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \\ & \left\{ w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, \\ & \left\{ w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), r), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T] \right\}, r \in \mathbb{R}, \end{aligned}$$

are uniformly integrable.

Proof: This is a simple modification of a well-known result which concerns uniform integrability of conditional expectations. We choose $\pi \in \mathcal{A}$.

Step 1: Let us consider the family

$$w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) = \mathbb{E} \left[e^{-\Gamma(R(\mathcal{T})) (X^\pi(\mathcal{T}) - J(\mathcal{T})\eta(P(\mathcal{T})))} \middle| \mathcal{F}_{\mathcal{T}} \right], \quad (5.17)$$

indexed with stopping times \mathcal{T} . We can observe that

$$\begin{aligned} & e^{-\Gamma(R(\mathcal{T})) (X^\pi(\mathcal{T}) - J(\mathcal{T})\eta(P(\mathcal{T})))} \\ & \leq e^{-(\gamma_0 + \gamma_1(-\infty)\epsilon) (X^\pi(\mathcal{T}) - J(\mathcal{T})\eta(P(\mathcal{T})))} \\ & \quad + e^{-(\gamma_0 + \gamma_1(+\infty)\epsilon) (X^\pi(\mathcal{T}) - J(\mathcal{T})\eta(P(\mathcal{T})))} := \mathcal{U}, \end{aligned} \quad (5.18)$$

where $\gamma_1(-\infty) = \sup_{r \in \mathbb{R}} \gamma_1(r)$ and $\gamma_1(+\infty) = \inf_{r \in \mathbb{R}} \gamma_1(r)$. From point 4 from Definition 3.1, we conclude that $\mathbb{E}[\mathcal{U}] < \infty$. We can establish the first property:

$$\begin{aligned} & \sup_{\mathcal{T}} \mathbb{E} [w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T}))] \\ & = \sup_{\mathcal{T}} \mathbb{E} \left[e^{-\Gamma(R(\mathcal{T})) (X^\pi(\mathcal{T}) - J(\mathcal{T})\eta(P(\mathcal{T})))} \right] \leq \mathbb{E}[\mathcal{U}] < \infty. \end{aligned} \quad (5.19)$$

Step 2: By Markov's inequality and (5.19), we derive the inequality

$$\begin{aligned} & Pr \left(w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C \right) \\ & \leq \frac{\mathbb{E} [w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T}))]}{C} \leq \frac{\mathbb{E}[\mathcal{U}]}{C}. \end{aligned}$$

Consequently, for any $\delta > 0$, we can choose a sufficiently large C such that

$$Pr \left(w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C \right) < \delta.$$

Step 3: Since the random variable \mathcal{U} defined in (5.18) is trivially uniformly integrable, then for any $\delta_0 > 0$, we can choose δ such that

$$Pr(A) < \delta \Rightarrow \mathbb{E}[\mathcal{U}\mathbf{1}_A] < \delta_0.$$

By Step 2, for any $\delta_0 > 0$, we can choose δ and C such that

$$\begin{aligned} Pr\left(w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C\right) &< \delta \Rightarrow \\ \mathbb{E}\left[\mathcal{U}\mathbf{1}_{w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C}\right] &< \delta_0. \end{aligned}$$

Step 4: By (5.17)-(5.18) and the property of conditional expectations, we get the inequality

$$\begin{aligned} \mathbb{E}\left[w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T}))\mathbf{1}_{w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C}\right] \\ \leq \mathbb{E}\left[\mathcal{U}\mathbf{1}_{w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C}\right]. \end{aligned}$$

Consequently, by Step 3, for any $\delta_0 > 0$, we can choose δ and C such that

$$\begin{aligned} Pr\left(w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C\right) &< \delta \Rightarrow \\ \mathbb{E}\left[w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T}))\mathbf{1}_{w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})) > C}\right] &< \delta_0. \end{aligned}$$

We conclude that the family $w^{J(\mathcal{T}),\pi}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T}))$ indexed with stopping times \mathcal{T} is uniformly integrable. The remaining families of random variables can be studied in the exactly the same way. \square

Proposition 5.2. *Let $\pi \in \mathcal{A}$. We consider functions $(\vartheta^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$, $(\varphi^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$ such that the families*

$$\begin{aligned} &\left\{\vartheta^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T]\right\}, \\ &\left\{\varphi^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), R(\mathcal{T})), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T]\right\}, \\ &\left\{\varphi^{J(\mathcal{T})}(\mathcal{T}, X^\pi(\mathcal{T}), P(\mathcal{T}), r), \mathcal{T} \text{ is an } \mathbb{F} - \text{stopping time}, \mathcal{T} \in [0, T]\right\}, r \in \mathbb{R}, \end{aligned}$$

are uniformly integrable, and $(\vartheta^k)_{k=0}^n, (\varphi^k)_{k=0}^n$ satisfy the PDEs:

$$\begin{aligned} &\vartheta_t^k(t, x, p) + \mathcal{L}_k^\pi \vartheta^k(t, x, p) - \mathcal{M}_k^\pi \varphi^k(t, x, p, x - kF(t, p)) \\ &+ \mathcal{L}_k^\pi \varphi^k(t, x, p, x - kF(t, p)) + \left(\vartheta^{k-1}(t, x - \beta(p), p) - \vartheta^k(t, x, p)\right)k\lambda \\ &+ \left(\varphi^{k-1}(t, x - \beta(p), p, x - kF(t, p)) \right. \\ &\quad \left. - \varphi^{k-1}(t, x - \beta(p), p, x - \beta(p) - (k-1)F(t, p))\right)k\lambda \\ &+ \Psi^k(t, x, p, x - kF(t, p)) = 0, \\ &\quad (t, x, p) \in [0, T] \times \mathbb{R} \times (0, \infty), \\ &\vartheta^k(T, x, p) = \Phi^k(x, p, x), \quad (x, p) \in \mathbb{R} \times (0, \infty), \end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
& \varphi_t^k(t, x, p, r) + \mathcal{L}_k^\pi \varphi^k(t, x, p, r) \\
& + \left(\varphi^{k-1}(t, x - \beta(p), p, r) - \varphi^k(t, x, p, r) \right) k\lambda + \Psi^k(t, x, p, r) = 0, \\
& (t, x, p) \in [0, T] \times \mathbb{R} \times (0, \infty), \quad r \in \mathbb{R}, \\
& \varphi^k(T, x, p, r) = \Phi^k(t, x, r), \quad (x, p) \in \mathbb{R} \times (0, \infty), \quad r \in \mathbb{R},
\end{aligned} \tag{5.21}$$

for $k \in \{0, 1, \dots, n\}$. Moreover, we assume that the functions $(\Psi^k)_{k=0}^n, (\Phi^k)_{k=0}^n$ satisfy the integrability conditions:

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T |\Psi^{J(s)}(s, X^\pi(s), P(s), r)| ds \right] < \infty, \quad r \in \mathbb{R}, \\
& \mathbb{E} \left[\int_0^T |\Psi^{J(s)}(s, X^\pi(s), P(s), R(s))| ds \right] < \infty, \\
& \mathbb{E} \left[|\Phi^{J(T)}(X^\pi(T), P(T), r)| \right] < \infty, \quad r \in \mathbb{R}, \\
& \mathbb{E} \left[|\Phi^{J(T)}(X^\pi(T), P(T), R(T))| \right] < \infty.
\end{aligned}$$

We have the representations:

$$\begin{aligned}
& \varphi^k(t, x, p, r) \\
& = \mathbb{E}_{t,x,p,k} \left[\Phi^{J(T)}(X^\pi(T), P(T), r) + \int_t^T \Psi^{J(s)}(s, X^\pi(s), P(s), r) ds \right], \\
& (t, x, p, k) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}, \quad r \in \mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
\vartheta^k(t, x, p) & = \varphi^k(t, x, p, x - kF(t, p)), \\
& (t, x, p, k) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, 1, \dots, n\}.
\end{aligned}$$

Proof: Let $(\tau_m)_{m=0}^\infty$ denote a localizing sequence of stopping times for (X^π, P, R) . We fix $(t, x, p, k, r) \in [0, T] \times \mathbb{R} \times (0, \infty) \times \{0, \dots, n\} \times \mathbb{R}$ and choose $\pi \in \mathcal{A}$. Applying Itô's formula to φ , with r fixed, and using equation (5.21), we can deduce that

$$\begin{aligned}
& \mathbb{E}_{t,x,p,k} [\varphi^{J(\tau_m)}(\tau_m, X^\pi(\tau_m), P(\tau_m), r)] - \varphi^k(t, x, p, r) \\
& = \mathbb{E}_{t,x,p,k} \left[- \int_t^{\tau_m} \Psi^{J(s)}(s, X^\pi(s), P(s), r) ds \right].
\end{aligned}$$

We take $\tau_m \rightarrow T$. Since the jumps of the process J are totally inaccessible, then $J(T-) = J(T)$, a.s.. By uniform integrability and dominated convergence theorem, we

derive that

$$\begin{aligned} \varphi^k(t, x, p, r) &= \mathbb{E}_{t,x,p,k} \left[-e^{-\Gamma(r)} (X^\pi(T) - J(T)\eta(P(T))) + \int_t^T \Psi^{J(s)}(s, X^\pi(s), P(s), r) ds \right]. \end{aligned}$$

Applying Itô's formula to ϑ and using equation (5.20), we can show that

$$\begin{aligned} &\mathbb{E}_{t,x,p,k} [\vartheta^{J(\tau_m)}(\tau_m, X^\pi(\tau_m), P(\tau_m))] - \vartheta^k(t, x, p) \\ &= \mathbb{E}_{t,x,p,k} \left[\int_t^{\tau_m} \left\{ \mathcal{M}_{J(s)}^\pi \varphi^{J(s)}(s, X^\pi(s), P(s), X^\pi(s) - J(s)F(s, P(s))) \right. \right. \\ &\quad - \mathcal{L}_{J(s)}^\pi \varphi^{J(s)}(s, X^\pi(s), P(s), X^\pi(s) - J(s)F(s, P(s))) \\ &\quad - \left(\varphi^{J(s)-1}(s, X^\pi(s) - \beta(P(s)), P(s), X^\pi(s) - J(s)F(s, P(s))) \right. \\ &\quad \left. \left. - \varphi^{J(s)-1}(s, X^\pi(s) - \beta(P(s)), P(s), X^\pi(s) - \beta(P(s)) - (J(s) - 1)F(s, P(s))) \right) J(s)\lambda \right. \\ &\quad \left. - \Psi^{J(s)}(s, X^\pi(s), P(s), X^\pi(s) - J(s)F(s, P(s))) \right\} ds \right]. \end{aligned}$$

Since the PDEs (5.21) also hold for $r = x - kF(t, p)$, we conclude that

$$\begin{aligned} &\mathbb{E}_{t,x,p,k} [\vartheta^{J(\tau_m)}(\tau_m, X^\pi(\tau_m), P(\tau_m))] - \vartheta^k(t, x, p) \\ &= \mathbb{E}_{t,x,p,k} \left[\int_t^{\tau_m} \left\{ \varphi_t^{J(s)}(s, X^\pi(s), P(s), X^\pi(s) - J(s)F(s, P(s))) \right. \right. \\ &\quad + \mathcal{M}_{J(s)}^\pi \varphi^{J(s)}(s, X^\pi(s), P(s), X^\pi(s) - J(s)F(s, P(s))) \\ &\quad + \left(\varphi^{J(s)-1}(s, X^\pi(s) - \beta(P(s)), P(s), X^\pi(s) - \beta(P(s)) - (J(s) - 1)F(s, P(s))) \right. \\ &\quad \left. \left. - \varphi^{J(s)}(s, X^\pi(s), P(s), X^\pi(s) - J(s)F(s, P(s))) \right) J(s)\lambda \right\} ds \right] \\ &= \mathbb{E}_{t,x,p,k} [\varphi^{J(\tau_m)}(\tau_m, X^\pi(\tau_m), P(\tau_m), R(\tau_m))] - \varphi^k(t, x, p, x - kF(t, p)), \end{aligned}$$

where the last term follows from Itô's formula applied to φ . We take $\tau_m \rightarrow T$. By uniform integrability, we arrive at $\vartheta^k(t, x, p) = \varphi^k(t, x, p, x - kF(t, p))$. \square

Proof of Theorem 5.1: First, we present detailed proofs of the assertions (iii) and (iv). At the end, we give a sketch of the proof for the assertions (i)-(ii). Let ϵ_0 denote a sufficiently small positive constant. We consider $\epsilon \in (0, \epsilon_0]$. By K we denote a constant which may change from line to line.

Step 1: We choose π_1 so that $\pi = \pi_0^* + \pi_1\epsilon \in \mathcal{B}$. By Proposition 5.1, $\pi \in \mathcal{A}$. Let $(v^{k,\pi})_{k=0}^n, (w^{k,\pi})_{k=0}^n$ denote the corresponding objective functions (3.4)-(3.5) for the optimization problem with the wealth-dependent risk aversion (3.3). By our assumption, $(v^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$ and $(w^{k,\pi})_{k=0}^n \in$

$\mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$. We will use the following four properties:

Step 1a: Since $v^{k,\pi}(t, x, p) = w^{k,\pi}(t, x, p, x - kF(t, p))$ by (3.6), then we have the following relations for the derivatives:

$$\begin{aligned}
v_x^{k,\pi}(t, x, p) &= w_x^{k,\pi}(t, x, p, x - kF(t, p)) + w_r^{k,\pi}(t, x, p, x - kF(t, p)), \\
v_{xp}^{k,\pi}(t, x, p) &= w_{xp}^{k,\pi}(t, x, p, x - kF(t, p)) + w_{rp}^{k,\pi}(t, x, p, x - kF(t, p)) \\
&\quad - w_{xr}^{k,\pi}(t, x, p, x - kF(t, p))kF_p(t, p) \\
&\quad - w_{rr}^{k,\pi}(t, x, p, x - kF(t, p))kF_p(t, p), \\
v_{xx}^{k,\pi}(t, x, p) &= w_{xx}^{k,\pi}(t, x, p, x - kF(t, p)) + w_{rr}^{k,\pi}(t, x, p, x - kF(t, p)) \\
&\quad + 2w_{xr}^{k,\pi}(t, x, p, x - kF(t, p)). \tag{5.22}
\end{aligned}$$

Step 1b: Since π_0^* is determined by (4.5) and $v_0^k(t, x, p) = w_0^k(t, x, p)$ (see (4.11)-(4.12)), we can also use the strategy

$$\pi_0^{k,*}(t, x, p) = -\frac{w_{0,x}^k(t, x, p)\mu + w_{0,px}^k(t, x, p)bp\sigma\rho}{w_{0,xx}^k(t, x, p)\sigma^2},$$

and the equation

$$w_{0,x}^k(t, x, p)\mu + w_{0,px}^k(t, x, p)bp\sigma\rho + \pi_0^{k,*}(t, x, p)w_{0,xx}^k(t, x, p)\sigma^2 = 0. \tag{5.23}$$

The terms on the left hand side of (5.23) can always be added to any equation without changing this equation. It is obvious that (5.23) also holds if we replace w_0^k with v_0^k .

Step 1c: We claim that the mapping $\epsilon \mapsto X^{\pi_0^* + \pi_1 \epsilon}(\cdot, \omega)$ is continuous in the topology of uniform convergence on $[0, \epsilon_0] \times [0, T]$ for *a.a.* ω . By Theorem V.7 from Protter (2005) and points 2-3 from Definition 5.1, there exists a unique solution $X^{\pi_0^* + \pi_1 \epsilon}$ to the SDE (3.1) for any $\epsilon \in [0, \epsilon_0]$. We have the dynamics:

$$\begin{aligned}
&d(X^{\pi_0^* + \pi_1 \epsilon}(t) - X^{\pi_0^* + \pi_1 \epsilon'}(t)) \\
&= \left\{ \pi_0^{J(t^-),*}(t, X^{\pi_0^* + \pi_1 \epsilon}(t), P(t)) - \pi_0^{J(t^-),*}(t, X^{\pi_0^* + \pi_1 \epsilon'}(t), P(t)) \right. \\
&\quad \left. + \epsilon \left(\pi_1^{J(t^-)}(t, X^{\pi_0^* + \pi_1 \epsilon}(t), P(t)) - \pi_1^{J(t^-)}(t, X^{\pi_0^* + \pi_1 \epsilon'}(t), P(t)) \right) \right. \\
&\quad \left. + (\epsilon - \epsilon') \pi_1^{J(t^-)}(t, X^{\pi_0^* + \pi_1 \epsilon'}(t), P(t)) \right\} \left(\mu dt + \sigma dW(t) \right).
\end{aligned}$$

Let us recall the continuous process H from points 2-3 of Definition 5.1 and we define the stopping times $\tau_n = \inf\{t \in [0, T] : H(t) \geq n\}$ for $n \in \mathbb{N}$. Standard estimates for SDEs lead us to the inequality

$$\mathbb{E} \left[\sup_{t \in [0, \tau_n]} |X^{\pi_0^* + \pi_1 \epsilon}(t) - X^{\pi_0^* + \pi_1 \epsilon'}(t)|^q \right] \leq K_{\epsilon_0, n} |\epsilon - \epsilon'|^q, \quad q \geq 2.$$

By Kolmogorov's lemma, our claim holds on $[0, \epsilon_0] \times [0, \tau_n]$ for *a.a.* ω . The continuity of the mapping $\epsilon \mapsto X^{\pi_0^* + \pi_1 \epsilon}(\cdot, \omega)$ on $[0, \epsilon_0] \times [0, T]$ for *a.a.* ω follows from the arguments from the proofs of Theorems V.7 and V.37 in Protter (2005).

Step 1d: We improve the estimates (5.11)-(5.14). Let us choose sufficiently small $q > 1, \kappa > 1, \iota > 1$, and let q^*, κ^*, ι^* denote their conjugates. We introduce the martingales:

$$\begin{aligned}\mathcal{M}(t) &= e^{-\gamma_0(X^{\pi_0^*}(t) - Y(t))}, \quad 0 \leq t \leq T, \\ M(t) &= e^{-\int_0^t q\kappa^* \epsilon \gamma_0 \pi_1(s) \sigma dW(s) - \frac{1}{2} \int_0^t |q\kappa^* \epsilon \gamma_0 \pi_1(s) \sigma|^2 ds}, \quad 0 \leq t \leq T.\end{aligned}$$

We note that

$$\left\| \int_0^T q\kappa^* \epsilon \gamma_0 \pi_1(s) \sigma dW(s) \right\|_{BMO} \leq K \left\| \int_0^T \pi_1(s) dW(s) \right\|_{BMO},$$

for all $\epsilon \in [0, \epsilon_0]$, and the constant K is independent of ϵ . By reverse Hölder inequality, for all $\epsilon \in [0, \epsilon_0]$ we can find a universal, sufficiently small $\iota > 1$ such that

$$\mathbb{E}[|M(t)|^\iota] \leq K,$$

where the constant K is independent of ϵ . Using the arguments from the proof of Proposition 5.1 together with Doob's inequality, we can now conclude that

$$\begin{aligned}\mathbb{E} \left[\sup_{t \in [0, T]} |e^{-\gamma_0 X^{\pi_0^* + \pi_1 \epsilon}(t)}|^q \right] \\ \leq K_1 \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{M}(t)|^{q\kappa} \right] \right)^{\frac{1}{\kappa}} \times \left(\mathbb{E} \left[\sup_{t \in [0, T]} |M(t)|^\iota \right] \right)^{\frac{1}{\kappa^* \iota}} \times \left(\mathbb{E} \left[e^{K_2 \epsilon^2 \int_0^T |\pi_1(s)|^2 ds} \right] \right)^{\frac{1}{\kappa^* \iota^*}} \\ \leq \frac{K_1}{\left(1 - K_2 \epsilon^2 \left\| \int_0^T \pi_1(s) dW(s) \right\|_{BMO}^2 \right)^{\frac{1}{r}}},\end{aligned}\tag{5.24}$$

with some $r > 1$, for all $\epsilon \in [0, \epsilon_0]$. The constants K_1, K_2 in (5.24) are independent of ϵ .

We also improve the estimate (5.16). Let us choose any $q > 1$. Applying Burkholder-Davis-Gundy inequality as in the proof of Lemma 5.1, we can show that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X^{\pi_0^* + \pi_1 \epsilon}(t)|^q \right] \leq K \left(1 + \epsilon^r \left\| \int_0^T \pi_1(s) dW(s) \right\|_{BMO}^r \right),\tag{5.25}$$

with some $r \geq 2$, for all $\epsilon \in [0, \epsilon_0]$. The constant K in (5.25) is independent of ϵ . We remark that the constants K in (5.24)-(5.25) depend on $\left\| \int_0^T \pi_0^*(s) dW(s) \right\|_{BMO}$. However, the dependence of constants on the applied strategies will not be pointed out if this dependence is not needed for the proof.

Step 2: Let us introduce the functions:

$$\begin{aligned} Q^k(t, x, p, r) &= w^{k,\pi}(t, x, p, r) - w_0^k(t, x, p) - w_1^k(t, x, p, r)\epsilon, \\ U^k(t, x, p) &= v^{k,\pi}(t, x, p) - v_0^k(t, x, p) - v_1^k(t, x, p)\epsilon. \end{aligned}$$

The functions quantify the approximation errors which we want to study. In this step we derive probabilistic representations for $(Q^k)_{k=0}^n$ and $(U^k)_{k=0}^n$. Since we assume that $(v^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$, $(w^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$ and Lemma 5.2 holds, we can apply Proposition 5.2 and derive PDEs for $(v^{k,\pi})_{k=0}^n, (w^{k,\pi})_{k=0}^n$. Using the PDEs (4.7)-(4.10) for $(v_0^k, v_1^k, w_0^k, w_1^k)_{k=0}^n$, we can next derive the PDEs for $(Q^k)_{k=0}^n, (U^k)_{k=0}^n$:

$$\begin{aligned} &Q_t^k(t, x, p, r) + \mathcal{L}_k^\pi Q^k(t, x, p, r) \\ &\quad + \left(Q^{k-1}(t, x - \beta(p), p, r) - Q^k(t, x, p, r) \right) k\lambda \\ &\quad + (\mathcal{L}_k^\pi - \mathcal{L}_k^{\pi_0^*})(w_0^k(t, x, p) + w_1^k(t, x, p, r)\epsilon) = 0, \\ &Q^k(T, x, p, r) = -e^{-\Gamma(r)(x-k\eta(p))} \\ &\quad + e^{\gamma_0(x-k\eta(p))} - \gamma_1(r)(x - k\eta(p))e^{-\gamma_0(x-k\eta(p))}\epsilon, \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} &U_t^k(t, x, p) + \mathcal{L}_k^\pi U^k(t, x, p) - \mathcal{M}_k^\pi Q^k(t, x, p, x - kF(t, p)) \\ &\quad + \mathcal{L}_k^\pi Q^k(t, x, p, x - kF(t, p)) \\ &\quad + \left(U^{k-1}(t, x - \beta(p), p) - U^k(t, x, p) \right) k\lambda \\ &\quad + \left(Q^{k-1}(t, x - \beta(p), p, x - kF(t, p)) \right. \\ &\quad \quad \left. - Q^{k-1}(t, x - \beta(p), p, x - \beta(p) - (k-1)F(t, p)) \right) k\lambda \\ &\quad + (\mathcal{L}_k^\pi - \mathcal{L}_k^{\pi_0^*})(v_0^k(t, x, p) + v_1^k(t, x, p)\epsilon) \\ &\quad - (\mathcal{M}_k^\pi - \mathcal{L}_k^\pi - \mathcal{M}_k^{\pi_0^*} + \mathcal{L}_k^{\pi_0^*})w_1^k(t, x, p, x - kF(t, p))\epsilon = 0 \\ &U^k(T, x, p) = -e^{-\Gamma(x-k\eta(p))(x-k\eta(p))} \\ &\quad + e^{\gamma_0(x-k\eta(p))} - \gamma_1(x - k\eta(p))(x - k\eta(p))e^{-\gamma_0(x-k\eta(p))}\epsilon. \end{aligned} \tag{5.27}$$

Recalling Definition 3.3, the strategy $\pi = \pi_0^* + \pi_1\epsilon$ and using (5.22)-(5.23), we can show

that

$$\begin{aligned}
& (\mathcal{L}_k^\pi - \mathcal{L}_k^{\pi_0^*})(w_0^k(t, x, p) + w_1^k(t, x, p, r)\epsilon) \\
&= \pi^k(t, x, p)(w_{0,x}^k(t, x, p) + w_{1,x}^k(t, x, p, r)\epsilon)\mu \\
&\quad - \pi_0^{k,*}(t, x, p)(w_{0,x}^k(t, x, p) + w_{1,x}^k(t, x, p, r)\epsilon)\mu \\
&\quad + \frac{1}{2}|\pi^k(t, x, p)|^2(w_{0,xx}^k(t, x, p) + w_{1,xx}^k(t, x, p, r)\epsilon)\sigma^2 \\
&\quad - \frac{1}{2}|\pi_0^{k,*}(t, x, p)|^2(w_{0,xx}^k(t, x, p) + w_{1,xx}^k(t, x, p, r)\epsilon)\sigma^2 \\
&\quad + \pi^k(t, x, p)(w_{0,px}^k(t, x, p) + w_{1,px}^k(t, x, p, r)b\rho\sigma\rho\epsilon)b\rho\sigma\rho \\
&\quad - \pi_0^{k,*}(t, x, p)(w_{0,px}^k(t, x, p) + w_{1,px}^k(t, x, p, r)b\rho\sigma\rho\epsilon)b\rho\sigma\rho \\
&\quad - \pi_1^k(t, x, p)\left(w_{0,x}^k(t, x, p)\mu \right. \\
&\quad \quad \left. + w_{0,px}^k(t, x, p)b\rho\sigma\rho + \pi_0^{k,*}(t, x, p)w_{0,xx}^k(t, x, p)\sigma^2\right)\epsilon \\
&= \pi_1^k(t, x, p)(w_{1,x}^k(t, x, p, r)\mu + w_{1,px}^k(t, x, p, r)b\rho\sigma\rho)\epsilon^2 \\
&\quad + \pi_0^{k,*}(t, x, p)\pi_1^k(t, x, p)w_{1,xx}^k(t, x, p, r)\sigma^2\epsilon^2 \\
&\quad + \frac{1}{2}|\pi_1^k(t, x, p)|^2(w_{0,xx}^k(t, x, p) + w_{1,xx}^k(t, x, p, r)\epsilon)\sigma^2\epsilon^2 \\
&:= \Psi^{k,\pi_1}(t, x, p, r), \tag{5.28}
\end{aligned}$$

and

$$\begin{aligned}
& (\mathcal{L}_k^\pi - \mathcal{L}_k^{\pi_0^*})(v_0^k(t, x, p) + v_1^k(t, x, p)\epsilon) \\
&\quad - (\mathcal{M}_k^\pi - \mathcal{L}_k^\pi - \mathcal{M}_k^{\pi_0^*} + \mathcal{L}_k^{\pi_0^*})w_1^k(t, x, p, x - kF(t, p))\epsilon \\
&= \Psi^{k,\pi_1}(t, x, p, x - kF(t, p)). \tag{5.29}
\end{aligned}$$

We investigate the function Ψ^{k,π_1} . We can calculate the derivatives of w_0^k, w_1^k since the explicit solutions (4.12), (4.14) are available. By the properties of $(h^k)_{k=0}^n, (g^k)_{k=0}^n$ specified in Proposition 4.1, point 3 of Definition 5.1 and (A6), we can derive the estimates:

$$\begin{aligned}
|w_{1,x}^k(t, x, p, r)| &\leq Ke^{-\gamma_0 x}(1 + |x|), & |w_{1,px}^k(t, x, p, r)| &\leq Ke^{-\gamma_0 x}(1 + |x|), \\
|w_{1,xx}^k(t, x, p, r)| &\leq Ke^{-\gamma_0 x}(1 + |x|), & |w_{0,xx}^k(t, x, p)| &\leq Ke^{-\gamma_0 x}, \\
|\pi_0^{k,*}(t, x, p)| &\leq K(1 + p),
\end{aligned}$$

which lead us to the estimate for the function Ψ^{k,π_1} :

$$\begin{aligned}
& |\Psi^{k,\pi_1}(t, x, p, r)| \\
& \leq K \left(|\pi_1^k(t, x, p)|^2 e^{-\gamma_0 x} + |\pi_1^k(t, x, p)| e^{-\gamma_0 x} (1 + |x|) (1 + p) \right) \epsilon^2 \\
& \quad + K |\pi_1^k(t, x, p)|^2 e^{-\gamma_0 x} (1 + |x|) \epsilon^3 \\
& \leq K e^{-\gamma_0 x} (1 + |x|) (1 + p)^2 (\epsilon^2 + \epsilon^3), \tag{5.30}
\end{aligned}$$

Applying Hölder's inequality and using (5.24)-(5.25) together with

$$\mathbb{E} \left[\sup_{t \in [0, T]} |P(t)|^q \right] < \infty, \quad \text{for all } q \geq 1, \tag{5.31}$$

we can deduce that

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T |\Psi^{J(s)}(s, X^\pi, P(s), r)| ds \right] < \infty, \\
& \mathbb{E} \left[\int_0^T |\Psi^{J(s)}(s, X^\pi, P(s), R(s))| ds \right] < \infty. \tag{5.32}
\end{aligned}$$

Finally, using the above results, Proposition 4.1, Lemmas 5.1-5.2 and the assumption that $(v^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty)) \cap \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R} \times (0, \infty))$, $(w^{k,\pi})_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$, we can apply Proposition 5.2 and establish probabilistic representations for the functions $(Q^k)_{k=0}^n, (U^k)_{k=0}^n$. We derive the key representation for the approximation error:

$$\begin{aligned}
U^k(t, x, p) &= \mathbb{E}_{t,x,p,k} \left[- e^{-\Gamma(x - k\eta(p))} (X^\pi(T) - J(T)\eta(P(T))) \right. \\
& \quad \left. + e^{\gamma_0 (X^\pi(T) - J(T)\eta(P(T)))} \right. \\
& \quad \left. - \gamma_1 (x - k\eta(p)) (X^\pi(T) - J(T)\eta(P(T))) e^{-\gamma_0 (X^\pi(T) - J(T)\eta(P(T)))} \right] \epsilon \\
& \quad + \mathbb{E}_{t,x,p,k} \left[\int_t^T \Psi^{J(s),\pi_1}(s, X^\pi(s), P(s), x - kF(t, p)) ds \right] \\
& := U_1^k(t, x, p) + U_2^k(t, x, p), \tag{5.33}
\end{aligned}$$

where $U_1^k(t, x, p)$ denotes the first expected value in (5.33), and $U_2^k(t, x, p)$ denotes the second expected value in (5.33).

Step 3: From now on, we explicitly consider $\pi = \pi_0^* + \pi_1^\delta \epsilon$, where π_1^δ is defined in (5.6). We choose π_1 so that $\pi_0^* + \pi_1^\delta \epsilon \in \mathcal{B}$. By Proposition 5.1 and point 3 from Definition 5.1, we note that

$$\begin{aligned}
& \left\| \int_0^T \pi_1^\delta(s) dW(s) \right\|_{BMO}^2 \\
& \leq 2 \left(\left\| \int_0^T \pi_1^*(s) dW(s) \right\|_{BMO}^2 + \left\| \int_0^T \pi_1(s) dW(s) \right\|_{BMO}^2 \right) \leq K, \tag{5.34}
\end{aligned}$$

for all $\delta \in [0, T - t]$. The constant K in (5.34) is independent of δ .

We study the first expected value in (5.33). Let $\gamma_1 := \gamma_1(x - k\eta(p))$. We investigate the random variable

$$\begin{aligned}
& -e^{-(\gamma_0 + \gamma_1 \epsilon)(X^\pi(T) - J(T)\eta(P(T)))} + e^{-\gamma_0(X^\pi(T) - J(T)\eta(P(T)))} \\
& \quad - \gamma_1(X^\pi(T) - J(T)\eta(P(T)))e^{-\gamma_0(X^\pi(T) - J(T)\eta(P(T)))}\epsilon \\
& = -\gamma_1^2 \int_0^1 |X^\pi(T) - J(T)\eta(P(T))|^2 e^{-\gamma_0(X^\pi(T) - J(T)\eta(P(T)))} \\
& \quad \times e^{-\gamma_1 \epsilon z(X^\pi(T) - J(T)\eta(P(T)))} (1 - z) dz \epsilon^2.
\end{aligned} \tag{5.35}$$

Let us choose $z \in [0, 1]$. As in the proof of Proposition 5.1 and Step 1d of this proof, we can observe that

$$\begin{aligned}
& e^{-(\gamma_0 + \gamma_1 \epsilon z)(X^\pi(T) - J(T)\eta(P(T)))} \\
& = e^{-(\gamma_0 + \gamma_1 \epsilon z)X^\pi(t) + \gamma_0 Y(t)} \mathcal{M}(T) M(T) \\
& \quad \times e^{\gamma_1 \epsilon z \left(\int_t^T J(s-) \alpha(P(s)) ds - \int_t^T \beta(P(s)) dJ(s) + J(T)\eta(P(T)) \right)},
\end{aligned}$$

where we introduce the strategy

$$\tilde{\pi}_z(s) = \gamma_1 z \pi_0^*(s) + (\gamma_0 + \gamma_1 \epsilon z) \pi_1^\delta(s), \quad t \leq s \leq T,$$

and the exponential martingales

$$\begin{aligned}
\mathcal{M}(s) & = e^{-\gamma_0(X^{\pi_0^*}(s) - X^{\pi_0^*}(t) - (Y(s) - Y(t)))}, \quad t \leq s \leq T, \\
M(s) & = e^{-\epsilon \left(\int_t^s \tilde{\pi}_z(u) \mu du + \int_t^s \tilde{\pi}_z(u) \sigma dW(u) \right)} \quad t \leq s \leq T.
\end{aligned}$$

By point 3 from Definition 5.1, Proposition 5.1, the properties (5.34) and (A6), we have the estimate

$$\left\| \int_0^T \tilde{\pi}_z(s) dW(s) \right\|_{BMO}^2 \leq K, \tag{5.36}$$

for all $z \in [0, 1]$, $\epsilon \in [0, \epsilon_0]$ and $\delta \in [0, T - t]$. Moreover, the constant K is independent of (z, ϵ, δ) .

We choose $q = 1$, or a sufficiently small $q > 1$, and a sufficiently small $\kappa > 1$. Using (5.34), (5.36) and the same arguments which led us to (5.11)-(5.14), (5.24)-(5.25)), we

can deduce the estimate

$$\begin{aligned}
& \mathbb{E}_{t,x,p,k} \left[\left| X^\pi(T) - J(T)\eta(P(T)) \right|^{2q} e^{-q(\gamma_0 + \gamma_1 \epsilon z) \left(X^\pi(T) - J(T)\eta(P(T)) \right)} \right] \\
& \leq \left(\mathbb{E}_{t,x,p,k} \left[\left| X^\pi(T) - J(T)\eta(P(T)) \right|^{2q\kappa^*} \right] \right)^{\frac{1}{\kappa^*}} \\
& \quad \times \left(\mathbb{E} \left[e^{-q\kappa(\gamma_0 + \gamma_1 \epsilon z) \left(X^\pi(T) - J(T)\eta(P(T)) \right)} \right] \right)^{\frac{1}{\kappa}} \\
& \leq \frac{K_1 \left(1 + \epsilon^{r_1} \left\| \int_0^T \pi_1^\delta(s) dW(s) \right\|_{BMO}^{r_1} \right)^{\frac{1}{r_2}}}{\left(1 - K_2 \epsilon^2 \left\| \int_0^T \tilde{\pi}_z(s) dW(s) \right\|_{BMO}^2 \right)^{\frac{1}{r_3}}} \leq K, \tag{5.37}
\end{aligned}$$

with some $r_1 \geq 2, r_2 > 1, r_3 > 1$, for all $z \in [0, 1], \epsilon \in [0, \epsilon_0], \delta \in [0, T - t]$. The constant K in (5.37) is independent of (ϵ, z, δ) .

By Fubini's theorem and (5.35)-(5.37), we can write

$$\begin{aligned}
U_1^k(t, x, p) &= -\gamma_1^2 \int_0^1 \mathbb{E}_{t,x,p,k} \left[\left| X^{\pi_0^* + \pi_1^\delta \epsilon}(T) - J(T)\eta(P(T)) \right|^2 \right. \\
& \quad \left. \cdot e^{-(\gamma_0 + \gamma_1 \epsilon z) \left(X^{\pi_0^* + \pi_1^\delta \epsilon}(T) - J(T)\eta(P(T)) \right)} \right] (1 - z) dz \epsilon^2. \tag{5.38}
\end{aligned}$$

From (5.37) we conclude that

$$|U_1^k(t, x, p)| \leq K \epsilon^2, \tag{5.39}$$

for all $\epsilon \in [0, \epsilon_0]$, and the constant K is independent of ϵ . By Step 1c, Lebesgue's dominated convergence theorem and uniform integrability (justified with (5.37)), we can prove the limit:

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \frac{U_1^k(t, x, p)}{\epsilon^2} \\
&= -\frac{1}{2} \gamma_1^2 \mathbb{E}_{t,x,p,k} \left[\left| X^{\pi_0^*}(T) - J(T)\eta(P(T)) \right|^2 e^{-\gamma_0 \left(X^{\pi_0^*}(T) - J(T)\eta(P(T)) \right)} \right], \tag{5.40}
\end{aligned}$$

where the right hand side of (5.40) only depends on π_0^* and is independent of π_1^δ (hence is independent of δ).

Step 4: We study the second expected value in (5.33). Recalling (5.28), we deal with

$$\Psi^{k,\pi_1}(t, x, p, r) = \Psi_1^{k,\pi_1}(t, x, p, r) \epsilon^2 + \Psi_2^{k,\pi_1}(t, x, p, r) \epsilon^3. \tag{5.41}$$

Let us choose $q = 1$, or a sufficiently small $q > 1$. Using the upper bound (5.30), the

estimates (5.24), (5.25), (5.31) and (5.34), we can deduce the estimate

$$\begin{aligned} & \mathbb{E}_{t,x,p,k} \left[\sup_{s \in [t,T]} |\Psi_1^{J(s),\pi_1}(s, X^{\pi_0^* + \pi_1^\delta \epsilon}(s), P(s), x - kF(t,p))|^q \right] \\ & \leq \frac{K_1 \left(1 + \epsilon^{r_1} \left\| \int_0^T \pi_1^\delta(s) dW(s) \right\|_{BMO}^{r_1} \right)^{\frac{1}{r_2}}}{\left(1 - K_2 \epsilon^2 \left\| \int_0^T \pi_1^\delta(s) dW(s) \right\|_{BMO}^2 \right)^{\frac{1}{r_3}}} \leq K, \end{aligned} \quad (5.42)$$

with some $r_1 \geq 2, r_2 > 1, r_3 > 1$, for all $\epsilon \in [0, \epsilon_0], \delta \in [0, T - t]$. The constant K is independent of (ϵ, δ) . We have the same estimate for Ψ_2^{k,π_1} . Consequently, we conclude that

$$|U_2^k(t, x, p)| \leq K\epsilon^2, \quad (5.43)$$

for all $\epsilon \in [0, \epsilon_0]$, and the constant K is independent of ϵ . By (5.41)-(5.42), we can also calculate the limit:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{U_2^k(t, x, p)}{\epsilon^2} \\ & = \mathbb{E}_{t,x,p,k} \left[\int_t^T \Psi_1^{J(s),\pi_1}(s, X^{\pi_0^*}(s), P(s), x - kF(t,p)) ds \right], \end{aligned} \quad (5.44)$$

where we use the property that $(w_0^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$, $(w_1^k)_{k=0}^n \in \mathcal{C}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2,2,2}([0, T] \times \mathbb{R} \times (0, \infty) \times \mathbb{R})$, point 2 from Definition 5.1 and similar arguments which led us to the limit (5.40). We note that the right hand side of (5.44) depends on π_0^* and π_1 , since Ψ_1^{k,π_1} depends on π_1 .

Step 5: Assertion (iii) follows from (4.1), (5.33), (5.39) and (5.43). We prove assertion (iv). Recalling (5.40), (5.44) and the definition of π_1^δ , we have to study the limit:

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(\mathbb{E}_{t,x,p,k} \left[\int_t^{t+\delta} \Psi_1^{J(s),\pi_1^*}(s, X^{\pi_0^*}(s), P(s), x - kF(t,p)) ds \right. \right. \\ & \quad \left. \left. - \int_t^{t+\delta} \Psi_1^{J(s),\pi_1}(s, X^{\pi_0^*}(s), P(s), x - kF(t,p)) ds \right] \right). \end{aligned}$$

By (5.42), Lebesgue's dominated convergence theorem and differentiation theorem, we derive

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(\mathbb{E}_{t,x,p,k} \left[\int_t^{t+\delta} \Psi_1^{J(s),\pi_1^*}(s, X^{\pi_0^*}(s), P(s), x - kF(t,p)) ds \right. \right. \\ & \quad \left. \left. - \int_t^{t+\delta} \Psi_1^{J(s),\pi_1}(s, X^{\pi_0^*}(s), P(s), x - kF(t,p)) ds \right] \right) \\ & = \Psi_1^{k,\pi_1^*}(t, x, p, x - kF(t,p)) - \Psi_1^{k,\pi_1}(t, x, p, x - kF(t,p)). \end{aligned} \quad (5.45)$$

Since $w_{0,xx}^k(t, x, p) < 0$ by (4.12), we can find π_1^k which maximizes $\Psi_1^{k,\pi_1}(t, x, p, x - kF(t, p))$. The optimal strategy takes the form

$$\begin{aligned} \tilde{\pi}_1^{k,*}(t, x, p) = & -\pi_0^*(t, x, p) \frac{w_{1,xx}^k(t, x, p, x - kF(t, p))}{w_{0,xx}^k(t, x, p, x - kF(t, p))} \\ & - \frac{w_{1,x}^k(t, x, p, x - kF(t, p))\mu + w_{1,px}^k(t, x, p, x - kF(t, p))b\rho\sigma}{w_{0,xx}^k(t, x, p, x - kF(t, p))\sigma^2}. \end{aligned}$$

Using (5.22) and (4.5), we can confirm (4.6). Consequently, the optimal $\tilde{\pi}_1^{k,*}$, which maximizes $\Psi_1^{k,\pi_1}(t, x, p, x - kF(t, p))$, is given by (4.18) and coincides with $\pi_1^{k,*}$. Hence, we conclude that

$$\Psi_1^{k,\pi_1^*}(t, x, p, x - kF(t, p)) - \Psi_1^{k,\pi_1}(t, x, p, x - kF(t, p)) \geq 0.$$

Since the limit (5.45) holds, the assertion (iv) is proved.

Step 6: We prove assertions (i)-(ii). We consider the PDEs (4.7) and (4.9) where we replace π_0^* with $\pi_0 \in \mathcal{B}$. We proceed in the same way as in Steps 1-4, and we can formally establish the zeroth-order expansion:

$$v^{k,\pi_0}(t, x, p) = V^{k,\pi_0}(t, x, p) + \mathcal{O}(\epsilon), \quad \epsilon \rightarrow 0,$$

where $V^{k,\pi}$ denotes the objective function for the time-consistent optimization problem (3.2) with $\Gamma(r) = \gamma_0$. By Remark 5.1.b, the strategy π_0^* is the optimal strategy for the time-consistent exponential utility maximization problem. Consequently, $V^{k,\pi_0}(t, x, p) \leq V^{k,\pi_0^*}(t, x, p)$, and the assertions are proved. \square

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