Indifference pricing of a life insurance portfolio with systematic mortality risk in a market with an asset driven by a Lévy process

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Abstract

In this paper we investigate the problem of pricing and hedging of life insurance liabilities. We consider a financial market consisting of a risk-free asset, with a constant rate of return, and a risky asset, whose price is driven by a Lévy process. We take into account a systematic mortality risk, and model mortality intensity as a diffusion process. The principle of equivalent utility is chosen as the valuation rule. In order to solve our optimization problems we apply techniques from the stochastic control theory. An exponential utility is considered in detail. We arrive at three pricing equations and investigate some properties of the premiums. An estimate of the finite-time ruin probability is derived. Indifference pricing with respect to a quadratic loss function is also briefly discussed.
Key words: Lévy process, stochastic mortality, counting process, Hamilton-Jacobi-Bellman equation, Lundberg's inequality.

1 Introduction

It is well-known that the price of any life insurance or pension product depends on demographic and financial assumptions. Traditional actuarial pricing principles (see [1, 2]) state that an actuary should set prudent (deterministic) estimates of future mortality rates and investment returns. However, insurance practice has shown that accurate, rather than prudent, estimates of future trends in mortality rates and investment returns are required. As precise forecasts are not possible, one should develop and apply probabilistic models to arrive at the best estimates.

In this paper we investigate the problem of pricing and hedging of life insurance liabilities in the case when the insurer can trade dynamically in a financial market. It is well-known (see for example [3, 4]) that a combined insurance and financial market is incomplete, in the sense that there does not exist a unique martingale measure which can be used to price a contract. In incomplete markets one would like to have a selection principle to reduce the class of possible martingale measures. An economically appealing method is the utility indifference principle, which is based on comparing the expected utilities of an investor when taking a risk and without it. Note that the set of equivalent martingale measures in a combined financial and insurance model, with a systematic mortality risk, is studied in [3, 4].

The utility indifference principle, in the case when an agent invests its wealth in a financial market, was first introduced in [5] and since then, it has become a popular method of pricing and hedging of financial risks in incomplete markets. This rule was extended in [6,7] to price insurance risks. In [6] explicit results are derived for an exponential utility function by solving the Hamilton-Jacobi equations, whereas in [7], and later in [8], mean-variance preferences are investigated and the Galtchouk-Kunita-Watanabe decomposition is applied in order to solve the problem.

In this paper we extend the results from [6]. We consider a financial market with a risky asset whose price dynamics is driven by a Lévy process, and we model the mortality intensity as a stochastic process of diffusion type. Both of these extensions are not only theoretically interesting but they are also of great practical relevance, as we explain later on. To the best our of knowledge, indifference pricing of life-insurance liabilities in this framework is taken up for the first time. We investigate three different types of indifference arguments. Two of them appear for the first time in the literature. The main goal of this paper is to derive reasonable pricing equations which can be easily applied by the insurer in the valuation process, together with hedging strategies. From the theoretical view point, our contribution consists in finding classical solutions of the corresponding Hamilton-Jacobi-Bellman equations.

One of the most important features of financial asset returns is their high variability, resulting from the heavy-tailed nature of empirical returns and observable large sudden movements in stock prices. The so-called six-standard deviation market moves are repeatedly seen in financial markets around the world. These properties rule out the possibility that the marginal distribution of an asset return is Gaussian. Moreover, all models of stock price dynamics which generate continuous sample paths are also inadequate. It is now well-known (see Chapters 1 and 7 in [9]) that Lévy processes can easily reproduce heavy tails, skewness and other distributional properties of asset returns, and, what is very important as well, can generate discontinuities in the price dynamics. As Lévy processes generate more realistic sample paths of stock prices, one should replace, in the celebrated Black-Scholes model, a Gaussian noise by a Lévy noise. One can expect that this would change significantly an investment strategy.

In the 1980's and 1990's mortality improvements turned out to be much greater than forecast, and the unexpected decrease in mortality rates affected the solvency of life-insurance companies and pension providers. Over the last 20 years mortality improvements have varied substantially and mortality rates have been evolving in a random fashion (see [10]). One can notice a general trend but there is still an unpredictable factor left which cannot be handled by any deterministic model. This is the reason why probabilistic models of mortality evolution have appeared in the literature (see [3, 11, 12, 13]). The possibility that the mortality intensity curve will evolve in a different way from that anticipated introduces an additional risk for the insurer. This risk, which is called the systematic mortality risk, cannot be diversified and the insurer is expected to charge a premium for this risk. Besides the systematic mortality risk, which is now gaining much attention, the insurer faces an unsystematic mortality risk, which was recognized long time ago. This risk arises when the actual number of deaths deviates from the anticipated number because of a finite number of policies in a portfolio. In contrast to the systematic mortality risk, the unsystematic mortality risk can be diversified by pooling. It is clear that both types of risk must be taken into account in order to avoid mispricing of life-insurance policies, which can have far-reaching consequences.

Very recently three papers have appeared dealing with the problem of pricing in the presence of the systematic mortality risk, (see [4, 14, 15]). A risk-minimizing criterion is applied in [4] to hedge a general payment process in a financial market consisting of a saving account and a bond. The dynamics of a short rate is given by an affine diffusion process, whereas the evolution of mortality intensity is described by a Cox-Ross-Ingersoll process. In the same model framework, the mean-variance indifference price for a pure endowment is also derived in [4]. In [14] a pure endowment contract is priced by assuming that the insurer is compensated for taking the risk via the Sharpe ratio of a portfolio consisting of a bond and the obligation to cover a claim. A hedging strategy is derived by minimizing the variance of the portfolio. Itô diffusion processes are used to describe the dynamics of a short rate and mortality intensity. The results from [14] are extended in [15] where an endowment insurance is priced. Let us point out again that our model framework seems to be a novelty. In addition to the three works cited above, it is worth mentioning that in [16] general valuation issues in incomplete markets are discussed, together with hedging of a longevity risk with longevity bonds.

This paper is structured as follows. In Section 2 we introduce our model of a financial market, a stochastic mortality intensity process and a payment process arising from a life insurance portfolio. Utility indifference pricing principles are discussed in Section 3. In Section 4 we investigate optimal control problems and derive the Hamilton-Jacobi-Bellman equations. An exponential utility function is considered in Section 5, where we arrive at explicit solutions. Some properties of premiums are studied in Section 5. A simple numerical example is also given in Section 5. Indifference pricing with a quadratic loss function is briefly discussed in Section 6. All proofs are given in the Appendix.

2 The model

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, where T denotes a finite and fixed time horizon. The filtration satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). The filtration $\mathbb{F}$ consists of three subfiltrations: $\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M \vee \mathbb{F}^N$, where $\mathbb{F}^F$ contains information about the financial market, $\mathbb{F}^M$ contains information about the mortality intensity and $\mathbb{F}^N$ contains information about the number of survivors in the portfolio. We assume that the subfiltrations $\mathbb{F}^F$ and $(\mathbb{F}^M, \mathbb{F}^N)$ are independent, which means that the future-life time of an insured is independent of the financial market. The measure $\mathbb{P}$ is the real-world, objective probability measure. All expected values are taken with respect to the measure $\mathbb{P}$, unless otherwise stated. The conditional expectation $\mathbb{E}[\cdot|X(t) = x, \lambda(t) = \lambda, N(t) = n]$ is denoted by $\mathbb{E}^{t,x,\lambda,n}[\cdot]$.

In the following subsections we introduce a financial market, a stochastic mortality intensity process and a life insurance portfolio.

2.1 The financial market

We consider a Lévy diffusion version of a Black-Scholes financial market. The price of a risk-free asset $B := (B(t), 0 \leq t \leq T)$ is described by an ordinary differential equation

$$\frac{dB(t)}{B(t)} = rdt, \quad B(0) = 1, \tag{2.1}$$

where r denotes the rate of interest. Another tradeable instrument in the market is a risky asset, and the dynamics of its price $S := (S(t), 0 \leq t \leq T)$ is given by a stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + \xi dL(t), \quad S(0) = 1, \tag{2.2}$$

where μ and ξ denote the drift and volatility, and $L := (L(t), 0 \leq t \leq T)$ denotes a zero-mean Lévy process, $\mathbb{F}^F$-adapted with $\mathbb{P}$-a.s. càdlàg sample paths (continuous on the right and having limits on the left). Let us recall that a Lévy process is a process with independent and stationary increments.

The process L is assumed to satisfy the following Lévy-Itô decomposition (see Chapter 2.4 in [17]):

$$L(t) = \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z\big(M(ds \times dz) - \nu(dz)ds\big), \tag{2.3}$$

where $W := (W(t), 0 \leq t \leq T)$ denotes a Brownian motion and $M((s,t] \times A) = \#\{s < u \leq t : (L(u) - L(u-)) \in A\}$ denotes a Poisson random measure, independent of W. We recall that the compensated measure $\tilde{M}((0,t] \times A) = M((0,t] \times A) - t\nu(A)$ is a martingale-valued measure, that is, $\tilde{M} := (\tilde{M}((0,t] \times A), 0 \leq t \leq T)$ is an $\mathbb{F}^F$-martingale for all Borel sets $A \in \mathcal{B}(\mathbb{R} - \{0\})$. The compensator ν is called a Lévy measure and satisfies $\int_{|z|<1} z^2 \nu(dz) < \infty$. For more information concerning Lévy processes, Lévy measures and Poisson random measures we refer the interested reader to [9, 17, 18].

Without loss of generality we may assume that $\xi = 1$. We also need the following assumption concerning the Lévy measure:

(A) the measure ν is defined on $(-1,\infty)$, with $\nu(\{0\}) = 0$, and satisfies the integrability condition $\int_{z\geq 1} z^2\nu(dz) < \infty$.

Notice that we do not exclude infinite active jump processes, for which the integral $\int_{|z|<1} |z|\nu(dz)$ is infinite.

Under the assumption **(A)** the stochastic differential equation (2.2) has a unique, positive and almost surely finite solution, given explicitly by the Doléans-Dade exponential,

$$
\begin{aligned}
S(t) &= \exp\Big\{\Big(\mu - \frac{1}{2}\sigma^2 + \int_{z>-1} (\log(1+z) - z)\nu(dz)\Big)t + \sigma W(t) \\
&\quad + \int_{(0,t]}\int_{z>-1} \log(1+z)\tilde{M}(ds\times dz)\Big\} \\
&= \exp\Big\{\mu_E t + \sigma W(t) + \int_{(0,t]}\int_{\mathbb{R}} z\tilde{M}_E(ds\times dz)\Big\}, \qquad (2.4)
\end{aligned}
$$

which is an exponential Lévy process with the measure $\nu_E(A) = \nu(\{z : \log(1+z) \in A\})$ (see Propositions 8.21 and 8.22 in [9]). We point out that in some papers, for example in [19], the exponential price model (2.4) serves as a starting point.

We also refer the reader to [9, 20], where different topics in financial modelling with Lévy processes are deeply investigated.

2.2 The mortality intensity

In order to capture the unexpected changes in mortality we assume that the mortality intensity, $\Lambda := (\lambda(t), 0 \leq t \leq T)$, is a stochastic process of diffusion type (an Itô diffusion process), with dynamics given by a stochastic differential equation

$$
d\lambda_x(t) = a(t,\lambda_x(t))dt + b(t,\lambda_x(t))d\bar{W}(t), \quad \lambda_x(0) = \lambda > 0, \qquad (2.5)
$$

where $\bar{W} := (\bar{W}(t), 0 \leq t \leq T)$ denotes an $\mathbb{F}^M$-adapted Brownian motion, independent of the Brownian motion W and the Poisson random measure $\tilde{M}$, with $\mathbb{P}$-a.s. continuous sample paths. We model, as usual, the mortality intensity as a function of two variables: the age x of an insured person at the time of issuing a policy, and the duration time t, since the issue. In the sequel the subscript x will be omitted.

We make the following assumptions concerning the stochastic mortality intensity process:

(B1) $a : [0,T]\times(0,\infty) \to \mathbb{R}, b : [0,T]\times(0,\infty) \to (0,\infty)$ are continuous functions, locally Lipschitz continuous in λ, uniformly in t,

(B2) there exists a sequence $(D_n)_{n\in\mathbb{N}}$ of bounded sets with $\bar{D}_n \subseteq (0,\infty)$ and $\bigcup_{n\geq 1} D_n = (0,\infty)$ such that the functions $a(t,\lambda)$ and $b^2(t,\lambda)$ are uniformly Lipschitz continuous on $[0,T]\times\bar{D}_n$,

(B3) $\mathbb{P}(\forall_{s\in[t,T]}\lambda(s) \in (0,\infty)|\lambda(t) = \lambda) = 1$ and $\sup_{s\in[t,T]} \mathbb{E}[|\lambda(s)|^2|\lambda(t) = \lambda] < \infty$ for all starting points $(t,\lambda) \in [0,T]\times(0,\infty)$.

We point out that diffusion dynamics of the intensity is used in all papers dealing with stochastic mortality (see [3, 4, 11, 12, 13]). It seems to be a reasonable assumption, as changes in mortality occur slowly and without sudden jumps. We remark that mortality intensity models, appearing in the literature, satisfy **(B1)-(B3)** and arise as special cases of (2.5).

Under the assumptions **(B1)** and **(B3)**, for each starting point $(t,\lambda) \in [0,T] \times (0,\infty)$, the mortality intensity process is nonexplosive on $[t,T]$ and there exists a unique strong solution to the stochastic differential equation (2.5) such that the mapping $(t,\lambda,s) \mapsto \lambda^{t,\lambda}(s)$ is $\mathbb{P}$-a.s. continuous (see [21, 22]). The assumption **(B2)** and the integrability condition in **(B3)** are required in order to verify the optimality of the derived investment strategy and to show that our candidate value function satisfies the Hamilton-Jacobi-Bellman equation in the classical sense.

We assume that the process Λ is $\mathbb{F}^M$-adapted, which is not a realistic assumption, since mortality intensity cannot be observed like the price of a tradeable asset. In this context, measurability means that it is possible to estimate the "true" intensity based on available data from a reference population. In this way we can treat mortality intensity as an observable quantity (see [3] for a discussion).

2.3 The life insurance portfolio

We deal with a portfolio consisting of identical life insurance policies issued at time 0 to a group of n_0 persons. Each policyholder is entitled to three types of payments. Firstly, there are amounts payable continuously at a rate c, as long as the insured person is alive, but no longer than for T years. These could be benefits, in the case of positive cashflows, or premiums, in the case of negative cashflows. Secondly, there is a benefit payable immediately at the time of death of the insured person, in the amount of b, within T years. Thirdly, there are two types of endowments: a certain initial premium $B(0)$ and a terminal survival benefit in the amount of $B(T)$, payable provided that the insured person is still alive at time T. The payment scheme, that we consider, obeys in all traditional life insurance and pension products (see [2]). Notice that we can deal not only with term and deffered life products but also with whole life products by considering the time T as the maximum possible future life-time of the insured person.

Let $T_1, T_2, ..., T_{n_0}$ denote the future life-times of the insured persons who are all at the same age x at the time of issuing the policies. We assume that the random variables $T_1, T_2, ..., T_{n_0}$ are identically distributed with the survival function

$$\mathbb{P}(T_i > t | \mathcal{F}_t^M) = e^{-\int_0^t \lambda(s)ds}, \quad i = 1, 2, ..., n_0. \tag{2.6}$$

The censored life-times $\big((T_1 \wedge T, \mathbf{1}\{T_1 \le T\}), ..., (T_{n_0} \wedge T, \mathbf{1}\{T_{n_0} \le T\})\big)$ are assumed to be $\mathbb{F}^N$-measurable. Notice that due to the process Λ, which affects the future life-times of all persons in the portfolio, the random variables $T_1, T_2, ..., T_{n_0}$ are correlated.

Let us consider the counting process $N := (N(t), 0 \le t \le T)$

$$N(t) = n_0 - \sum_{i=1}^{n_0} \mathbf{1}\{T_i \le t\}, \tag{2.7}$$

which counts the number of survivors in the portfolio. Our main object of interest is the cumulative payment process $P := (P(t), 0 \le t \le T)$ with the dynamics

$$dP(t) = N(t-)cdt - bdN(t) + N(T)B(T)d\mathbf{1}\{t \ge T\}, \tag{2.8}$$

with $P(0) = -n_0 B(0)$. For more information on modelling a life insurance portfolio and its cumulative payment stream with a point process, we refer the reader to Chapter 1 in [23]. We point out that the process P is $\mathbb{F}^N$-adapted with $\mathbb{P}$-a.s. càdlàg sample paths of finite variations.

We remark that in [4] the same payment process is considered, whereas in [14, 15], a portfolio consisting of only pure endowments and term insurance policies is investigated.

3 Indifference pricing principle

In this section we discuss indifference valuation rules which may be applied when dealing with pricing of life insurance liabilities. Before we state the principles we first introduce a wealth process and some notations which we will use in the sequel.

Consider the wealth process of the insurer $X^\pi := (X^\pi(t), 0 \le t \le T)$ who handles the payment process P arising from the issued policies. The dynamics of the process X^π is given by the stochastic differential equation

$$\begin{aligned} dX^\pi(t) &= \pi(t)\Big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt\times dz)\Big) \\ &\quad + \big(X^\pi(t-) - \pi(t)\big) r dt - dP(t), \\ X(0) &= x - P(0), \end{aligned} \tag{3.1}$$

where $\pi(t)$ denotes the amount of the wealth invested in the risky asset and x denotes the initial wealth of the insurer.

We define a family of processes, which we find useful when deriving the Hamilton-Jacobi-Bellman equation. For $n = 0, 1, ..., n_0$ let $X_n^{\pi_n} := (X_n^{\pi_n}(t), 0 \le t \le T)$ denote the process with dynamics given by the stochastic differential equation

$$\begin{aligned} dX_n^{\pi_n}(t) &= \pi_n(t)\Big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt\times dz)\Big) \\ &\quad + \big(X_n^{\pi_n}(t-) - \pi_n(t)\big) r dt - ncdt. \end{aligned} \tag{3.2}$$

Notice that the equation (3.2), with the control π_n, describes the evolution of the process X^π, when there are n policies in the portfolio. We denote by X_n^{s,x,π_n} the process which starts at $X(s) = x$ and whose dynamics is given by (3.2) for $t > s$. Let the stopping time τ_i, for $i = 0, 1, ..., n_0 - 1$, denote the time of the $(n_0 - i)$th death. We set $\tau_{n_0} = 0$. The process X^π can be defined recursively as

$$X^\pi(t) = \begin{cases} x + n_0 B(0), & t = 0, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(t), & \tau_n < t < \tau_{n-1} \wedge T, \tau_n < T, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(\tau_{n-1}) - b, & t = \tau_{n-1}, \tau_{n-1} < T, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(T) - b - (n-1)B(T), & t = T, \tau_{n-1} = T, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(T) - nB(T), & t = T, \tau_{n-1} > T, \\ X_0^{\tau_0, X^\pi(\tau_0), \pi_0}(t), & \tau_0 < t \le T, \tau_0 < T. \end{cases} \tag{3.3}$$

The above recursion starts at $n = n_0$.

We will also deal with the discounted wealth process $Y^\pi := (e^{-\rho t} X^\pi(t), 0 \le t \le T)$. The following relation holds:

$$Y^{\pi}(t)=\begin{cases} y+n_0B(0), & t=0,\\ Y_n^{\tau_n,Y^{\pi}(\tau_n),\pi_n}(t), & \tau_n<t<\tau_{n-1}\wedge T, \tau_n<T,\\ Y_n^{\tau_n,Y^{\pi}(\tau_n),\pi_n}(\tau_{n-1})-be^{-\rho\tau_{n-1}}, & t=\tau_{n-1}, \tau_{n-1}<T,\\ Y_n^{\tau_n,Y^{\pi}(\tau_n),\pi_n}(T)-be^{-\rho T}-(n-1)B(T)e^{-\rho T}, & t=T, \tau_{n-1}=T,\\ Y_n^{\tau_n,Y^{\pi}(\tau_n),\pi_n}(T)-nB(T)e^{-\rho T}, & t=T, \tau_{n-1}>T\\ Y_0^{\tau_0,Y^{\pi}(\tau_0),\pi_0}(t), & \tau_0<t\le T, \tau_0<T, \end{cases} \tag{3.4}$$

where the dynamics of $Y_n^{\pi_n}$ is described by the equation

$$\begin{aligned} dY_n^{\pi_n}(t) &= e^{-\rho t}\pi_n(t)\Big((\mu-r)ds+\sigma dW(t)+\int_{z>-1} z\tilde{M}(dt\times dz)\Big)\\ &\quad +Y_n^{\pi_n}(t-)(r-\rho)dt-nce^{-\rho t}dt. \end{aligned} \tag{3.5}$$

Let u denote a function describing the preferences of the insurer. In the classical decision-making theory under uncertainty, u is assumed to be a utility function (concave and increasing). However, other objective functions are also reasonable; for example in [4, 7, 8] mean-variance preferences are considered. Under the indifference principle the insurer, with the initial wealth x, prices the payment process P in such a way, that he is indifferent, with respect to his preferences u, between writing or not writing n_0 life insurance contracts. According to this rule the insurer arranges the payment process so that

$$\begin{aligned} &\sup_{\pi}\mathbb{E}\big[u(X^{\pi}(T))|X(0)=x+n_0B(0), \lambda(0)=\lambda, N(0)=n_0\big]\\ &\qquad =\sup_{\pi_0}\mathbb{E}\big[u(X_0^{\pi_0}(T))|X_0(0)=x\big]. \end{aligned} \tag{3.6}$$

Under the valuation rule (3.6) the insurer maximizes the utility of his wealth at the fixed terminal time T. However, the life insurance portfolio may terminate before time T and the insurer might be interested in valuating the utility of the return on its insurance portfolio only up to the time when the portfolio terminates, without any additional financial gains which may occur after covering the final payment. In [6] an indifference pricing with respect to a random time, which is the time of death of an insured person, is proposed. We continue to investigate this principle and deal with the following valuation rule:

$$\begin{aligned} &\sup_{\pi}\mathbb{E}\big[u(Y^{\pi}(\tau_0\wedge T))|Y(0)=y+n_0B(0), \lambda(0)=\lambda, N(0)=n_0\big]\\ &\qquad = u(y). \end{aligned} \tag{3.7}$$

Under this principle the insurer prices the payment process in such a way that the expected utility of wealth at the time of the termination of the portfolio, discounted at the time when the portfolio is issued, equals the utility of wealth available before issuing the policies. We point out that indifference arguments with respect to a discounted wealth process have already been applied in [4,7,8]. We can as well consider the following pricing principle:

$$\begin{aligned} &\sup_{\pi}\mathbb{E}\big[u(Y^{\pi}(\tau_0\wedge T))|Y(0)=y+n_0B(0), \lambda(0)=\lambda, N(0)=n_0\big]\\ &\qquad =\sup_{\pi_0}\mathbb{E}\big[u(Y_0^{\pi_0}(\tau_0\wedge T))|Y(0)=y, \lambda(0)=\lambda\big], \end{aligned} \tag{3.8}$$

which extends the right side of (3.7) by taking into account a trading strategy. We remark that the principle (3.8), with $\rho=r=0$, is considered in [6].

We believe that the valuation rules (3.7) and (3.8) are reasonable. In section 5 we show that for an exponential utility, the pricing equations (3.7) and (3.8) are interesting generalizations of the equation (3.6).

4 The optimal stochastic control problems

In this section we derive the corresponding Hamilton-Jacobi-Bellman equations and a classical verification theorem.

First we investigate the pricing principle (3.6). Let us define optimal value functions

$$V_n(t,x,\lambda) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[u(X^{\pi}(T))|X(t)=x,\lambda(t)=\lambda,N(t)=n\big], \tag{4.1}$$

for $n = 0,1,...,n_0$. Notice that $V_0(t,x,\lambda) = V_0(t,x)$ is the value function for the standard investment problem of maximizing the expected utility of wealth at the terminal time, without any payment process. This optimization problem is well-understood (see Chapter 3 in [24]). We have to find the value function V_n for $n > 0$.

The following lemma, which is intuitively clear, turns out to be useful in the derivation of the solution. We use the family of processes defined in Section 3.

Lemma 4.1. *For $n = 1,2,...,n_0$ and all $(t,x,\lambda) \in [0,T]\times\mathbb{R}\times(0,\infty)$ the value function $V_n(t,x,\lambda)$ has the following representation:*

$$\begin{aligned} V_n(t,x,\lambda) \;&=\; \sup_{\pi_n\in\mathcal{A}} \mathbb{E}^{t,x,\lambda,n}\big[u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\} \\ &\quad + V_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b)\mathbf{1}\{\tau_{n-1} \le T\}\big], \end{aligned} \tag{4.2}$$

provided that the value function V_{n-1} is well-defined.

This lemma is interesting as it gives a method of solving the optimization problem (4.1) for the portfolio consisting of $n = n_0$ policies. First, one has to find the value function V_0 and then solve the optimization problems (4.2) for $n = 1,2,...,n_0$ consecutively. Each optimization problem in this iterative procedure involves the value function from the previous step. The optimal investment strategy $\hat{\pi}$ consists of a sequence of n_0+1 controls, $\hat{\pi} = (\hat{\pi}_{n_0},...,\hat{\pi}_0)$, where $\hat{\pi}_0$ is the optimal investment strategy for the standard optimization problem, whereas the other $\hat{\pi}_n$ are the optimal investment strategies derived by solving the problems (4.2) iteratively.

Let us introduce a set of admissible strategies and two operators.

Definition 4.1. *The sequence of controls $\pi_n := (\pi_n(t), 0 < t \le T)$ is an admissible strategy, $\pi_n \in \mathcal{A}$, if it satisfies the following assumptions:*

1. *$\pi_n : (0,T]\times\Omega \mapsto \mathbb{R}$ is a predictable mapping with respect to the filtration $\mathbb{F}$,*
2. *$\int_0^T \pi_n^2(t)dt < \infty$ $\mathbb{P}$-a.s.,*
3. *the stochastic differential equation (3.2) has a unique solution $X_n^{\pi_n}$,*

for all $n = 0,1,...,n_0$.

We conclude that for any $\pi_n \in \mathcal{A}$ the process $X_n^{\pi_n}$ is a semimartingale with càdlàg sample paths (see Chapter 4.3.3 in [17]).

Definition 4.2. *Let $\mathcal{L}_{F,\rho}$ denote the integro-differential operator given by*

$$\begin{aligned} \mathcal{L}_{F,\rho}^{\pi} h(t,x) &= \big(e^{-\rho t}\pi(\mu - r) + x(r-\rho)\big)\frac{\partial h}{\partial x}(t,x) + \frac{1}{2}e^{-2\rho t}\pi^2\sigma^2\frac{\partial^2 h}{\partial x^2}(t,x) \\ &\quad + \int_{z>-1}\Big(h(t, x+e^{-\rho t}\pi z) - \phi(t,x) - e^{-\rho t}\pi z\frac{\partial h}{\partial x}(t,x)\Big)\nu(dz), \end{aligned} \tag{4.3}$$

and let $\mathcal{L}_M$ denote the differential operator given by

$$\mathcal{L}_M h(t,\lambda) = a(t,\lambda)\frac{\partial h}{\partial \lambda}(t,\lambda) + \frac{1}{2}b^2(t,\lambda)\frac{\partial^2 h}{\partial \lambda^2}(t,\lambda). \tag{4.4}$$

These two operators are defined for functions h such that the expressions $\mathcal{L}h$ are pointwise well-defined and all derivatives appearing in $\mathcal{L}h$ exist and are continuous functions.

Below we state a classical verification theorem. Let us simply denote the operator $\mathcal{L}_{F,0}$ by $\mathcal{L}_F$.

Theorem 4.1. *Assume that v_{n-1} is a candidate function, which coincides with the optimal value function V_{n-1}, such that*

$$\mathbb{E}^{0,x,\lambda,n}\left[\int_0^T \big|v_{n-1}(t, X_n^{\pi_n}(t), \lambda(t))\big|^2 dt\right] < \infty, \tag{4.5}$$

for $\pi_n \in \mathcal{A}$. Let $v_n \in \mathcal{C}^{1,2,2}([0,T) \times \mathbb{R} \times (0,\infty)) \cap \mathcal{C}([0,T] \times \mathbb{R} \times (0,\infty))$ satisfy, for $\pi_n \in \mathcal{A}$,

$$\begin{aligned} 0 &\geq \frac{\partial v_n}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\pi_n} v_n(t,x,\lambda) - nc\frac{\partial v_n}{\partial x}(t,x,\lambda) + \mathcal{L}_M v_n(t,x,\lambda) \\ &\quad + n\lambda\big(v_{n-1}(t, x-b, \lambda) - v_n(t,x,\lambda)\big), \end{aligned} \tag{4.6}$$

for all $(t,x,\lambda) \in [0,T) \times \mathbb{R} \times (0,\infty)$, with

$$v_n(T,x,\lambda) = u(x - nB(T)), \quad \forall (x,\lambda) \in \mathbb{R} \times (0,\infty). \tag{4.7}$$

Assume also that for $\pi_n \in \mathcal{A}$,

$$\begin{aligned} \mathbb{E}^{0,x,\lambda,n}\bigg[\int_0^T \int_{z>-1} &\big|v_n(t, X_n^{\pi_n}(t-) + \pi_n(t)z, \lambda(t)) \\ &- v_n(t, X_n^{\pi_n}(t-), \lambda(t))\big|^2 \nu(dz)dt\bigg] < \infty, \end{aligned} \tag{4.8}$$

$$\mathbb{E}^{0,x,\lambda,n}\left[\int_0^T \big|v_n(t, X_n^{\pi_n}(t), \lambda(t))\big|^2 dt\right] < \infty, \tag{4.9}$$

and

$$\{v_n^-(\mathcal{T}, X_n^{\pi_n}(\mathcal{T}), \lambda(\mathcal{T}))\}_{0<\mathcal{T}\leq T} \text{ is uniformly integrable for all stopping times } \mathcal{T}. \tag{4.10}$$

Then

$$v_n(t,x,\lambda) \geq V_n(t,x,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty). \tag{4.11}$$

If additionally there exists an admissible control $\hat{\pi}_n \in \mathcal{A}$ such that

$$\begin{aligned} 0 &= \frac{\partial v_n}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\hat{\pi}_n} v_n(t,x,\lambda) - nc\frac{\partial v_n}{\partial x}(t,x,\lambda) + \mathcal{L}_M v_n(t,x,\lambda) \\ &\quad + n\lambda\big(v_{n-1}(t,x-b,\lambda) - v_n(t,x,\lambda)\big), \end{aligned} \tag{4.12}$$

for all $(t,x,\lambda) \in [0,T) \times \mathbb{R} \times (0,\infty)$, and

$$\{v_n(\mathcal{T}, X_n^{\hat{\pi}_n}(\mathcal{T}), \lambda(\mathcal{T}))\}_{0<\mathcal{T}\le T} \text{ is uniformly integrable for all stopping times } \mathcal{T}, \tag{4.13}$$

then

$$v_n(t,x,\lambda) = V_n(t,x,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty), \tag{4.14}$$

and $\hat{\pi}_n$ is the optimal control for the optimization problem (4.2).

To claim that a solution found is optimal, one has to verify that it satisfies all conditions stated in the verification theorem. The conditions from Theorem 4.1 can be relaxed and one can consider a solution in the viscosity sense. As we are able to find smooth solutions, we do not investigate this concept.

Consider now the pricing principles (3.7) and (3.8). Let us define the optimal value functions

$$W_n(t,x,\lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\big[u(Y^\pi(\tau_0 \wedge T)) | Y(t) = y, \lambda(t) = \lambda, N(t) = n\big], \tag{4.15}$$

for $n = 1, 2, ..., n_0$. In view of equations (3.4) and (3.5) and Lemma 4.1 we can arrive at the Hamilton-Jacobi-Bellman equation corresponding to the optimization problem (4.15):

$$\begin{aligned} 0 &= \frac{\partial w_n}{\partial t}(t,y,\lambda) + \sup_{\pi_n \in \mathbb{R}} \big\{\mathcal{L}_{F,\rho}^{\pi_n} w_n(t,y,\lambda)\big\} - nce^{-\rho t}\frac{\partial w_n}{\partial x}(t,y,\lambda) + \mathcal{L}_M w_n(t,y,\lambda) \\ &\quad + n\lambda\big(w_{n-1}(t, y - be^{-\rho t}, \lambda) - w_n(t,y,\lambda)\big), \end{aligned} \tag{4.16}$$

with the terminal condition

$$w_n(T,y,\lambda) = u(y - nB(T)e^{-\rho T}). \tag{4.17}$$

Notice that this time $w_0(t,x,\lambda) = u(x)$, as the optimization procedure stops at time $\tau_0 \wedge T$. The reader can easily modify the conditions in Theorem 4.1 for the candidate value function w_n.

To the best of our knowledge the optimization problems (4.1), (4.2) and (4.15), together with Lemma 4.1 and the verification theorem, are new in the financial and actuarial literature. In the life-time portfolio selection theory the representation (4.2) means that the investor faces the terminal utility u and the bequest utility v_{n-1}. We point out that the problem of maximizing the utility with respect to a random time is investigated in [25]. Most of the results there deal with a deterministic intensity of exiting the market. One example considers a stochastic density process, modelled as a geometric Brownian motion, and the solution is derived for power utility functions.

We find explicit solutions of the Hamilton-Jacobi-Bellman equations (4.12) and (4.16) for an exponential utility and a quadratic loss function.

5 Exponential utility function

In this section we assume that the insurer applies an exponential utility function of the form $u(x) = -\frac{1}{\alpha}e^{-\alpha x}, \alpha > 0$. We point out that exponential indifference pricing is well-known in insurance theory (see [1, 2, 6, 19]).

5.1 Indifference pricing with respect to terminal time T

The first step in the iterative procedure (4.2) is to find the value function v_0. The following lemma is taken from [26].

Lemma 5.1. *Consider the Hamilton-Jacobi-Bellman equation associated with the optimization problem (4.1) for* $n = 0$*:*

$$\begin{aligned} 0 &= \frac{\partial v_0}{\partial t}(t,x) + \sup_{\pi_0 \in \mathbb{R}} \left\{ \mathcal{L}_F^{\pi_0} v_0(t,x) \right\}, \quad (t,x) \in [0,T) \times \mathbb{R}, \\ v_0(T,x) &= -\frac{1}{\alpha} e^{-\alpha x}, \quad x \in \mathbb{R}. \end{aligned} \tag{5.1}$$

The function v_0 *defined as*

$$v_0(t,x) = -\frac{1}{\alpha} e^{-\alpha f(t) x + g(t)} \tag{5.2}$$

satisfies the above equation in the classical sense, with $f(t) = e^{r(T-t)}$ *and* $g(t) = G(\hat{\kappa})(T-t)$*, where* $\hat{\kappa}$ *is the unique minimizer of the convex function*

$$G(\kappa) = -\kappa(\mu - r) + \frac{1}{2}\kappa^2\sigma^2 + \int_{z>-1} (e^{-\kappa z} - 1 + \kappa z)\nu(dz). \tag{5.3}$$

The optimal investment strategy is $\hat{\pi}_0(t) = \frac{\hat{\kappa}}{\alpha f(t)}$*.*

In [19] the sensitivity of the optimal investment strategy with respect to jump size, jump activity and jump asymmetry is investigated. The conclusion is that jumps can significantly change the optimal amount which should be invested in the risky asset and this amount should be smaller than in the case of lognormally distributed returns.

In this section we assume that $\mu > r$. Under this assumption $\hat{\kappa} > 0$ and $\hat{\pi}_0 > 0$, so the risky asset is never short-sold. Notice that the function G is strictly decreasing on the right of zero, which implies that $G(\hat{\kappa}) < 0$.

We postulate the following relation:

$$v_n(t,x,\lambda) = v_0(t,x)\phi_n(t,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty), \tag{5.4}$$

with $\phi_0(t,\lambda) = 1$ for all $(t,\lambda) \in [0,T] \times (0,\infty)$. Assume that we have found the value function v_{n-1} (or equivalently ϕ_{n-1}), together with the optimal investment strategy $\hat{\pi}_{n-1}$ applied when there are $n-1$ policies in the portfolio. The next step is to solve the optimization problem (4.2) and find the function ϕ_n and the optimal investment strategy $\hat{\pi}_n$ which is applied when there are n policies in the portfolio. By substituting (5.4) into (4.12) we arrive at

$$\begin{aligned} 0 \;=\; & v_0(t,x)\frac{\partial \phi_n}{\partial t}(t,\lambda) + \phi_n(t,\lambda)\left(\frac{\partial v_0}{\partial t}(t,x) + \sup_{\pi_n \in \mathbb{R}} \mathcal{L}_F^{\pi_n} v_0(t,x) \right) \\ & + v_0(t,x)\alpha f(t) n c \phi_n(t,\lambda) + v_0(t,x)\mathcal{L}_M \phi_n(t,\lambda) \\ & + v_0(t,x) n\lambda \big(e^{\alpha f(t) b}\phi_{n-1}(t,\lambda) - \phi_n(t,\lambda) \big). \end{aligned} \tag{5.5}$$

Notice that the strategy π_n, which realizes the supremum in (5.5), is exactly the same as the strategy π_0 realizing the supremum in (5.1). This yields an important statement that the optimal investment strategy, which should be applied when there aren policies in the portfolio, is independent of n and equals the optimal investment strategy when there is no insurance risk:

$$\hat{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha} e^{-r(T-t)}, \quad n = 0, 1, ..., n_0. \tag{5.6}$$

One could have expected such a result. As our objective is to maximize the expected wealth by investing in the financial market, it seems that there is no reason why the strategy, that achieves this goal, should depend on the payment process which is independent of the financial market.

From the equation (5.5) we derive a partial differential equation for the function $\phi_n(t, \lambda)$:

$$\begin{aligned} 0 &= \frac{\partial \phi_n}{\partial t}(t, \lambda) + \mathcal{L}_M \phi_n(t, \lambda) + (\alpha f(t) nc - n\lambda)\phi_n(t, \lambda) \\ &+ n\lambda e^{\alpha f(t) b}\phi_{n-1}(t, \lambda), \quad (t, \lambda) \in [0, T) \times (0, \infty), \\ &\phi_n(T, \lambda) = e^{\alpha n B(T)}. \end{aligned} \tag{5.7}$$

One can prove the following lemma.

Lemma 5.2. *Assume that $\phi_{n-1}(t, \lambda) \in \mathcal{C}^{1,2}([0, T) \times (0, \infty)) \cap \mathcal{C}_b([0, T] \times (0, \infty))$. Then the equation (5.7) has a unique solution $\phi_n(t, \lambda) \in \mathcal{C}^{1,2}([0, T) \times (0, \infty)) \cap \mathcal{C}_b([0, T] \times (0, \infty))$. The following probabilistic representation holds:*

$$\begin{aligned} \phi_n(t, \lambda) &= \mathbb{E}^{t,\lambda}\Big[e^{\alpha n B(T)} e^{\int_t^T (\alpha f(s) nc - n\lambda(s)) ds} \\ &\quad + \int_t^T e^{\alpha f(s) b} n\lambda(s)\phi_{n-1}(s, \lambda(s)) e^{\int_t^s (\alpha f(u) nc - n\lambda(u)) du} ds\Big]. \end{aligned} \tag{5.8}$$

The representation (5.8) is known as the Feynman-Kac formula. Notice that $\phi_0 = 1$ satisfies the assumption of Lemma 5.2 so that we can solve the partial differential equations recursively to obtain the sequence of functions $\phi_n(t, \lambda) \in \mathcal{C}^{1,2}([0, T) \times (0, \infty)) \cap \mathcal{C}_b([0, T] \times (0, \infty))$ for all $n = 1, 2, ..., n_0$. This completes the derivation of the solution to our optimization problem.

The recursion (5.8) is not very appealing from the computational point of view. The next lemma not only gives an explicit representation of the function ϕ_{n_0} but also provides an interesting insight into the structure of our solution.

Lemma 5.3. *The function $\phi_{n_0}(0, \lambda)$ can be represented as*

$$\phi_{n_0}(0, \lambda) = \mathbb{E}^{0,\lambda}\left[\exp\left(\alpha \int_{(0,T]} e^{r(T-t)} dP(t)\right)\right]. \tag{5.9}$$

We remark that this simple representation has not been noticed in [6] in the case of deterministic mortality.

By recalling the indifference pricing principle (3.6) we can now state the equation

$$n_0 B(0) = \frac{1}{\alpha} e^{-rT} \log \mathbb{E}^{0,\lambda}\left[\exp\left(\alpha \int_{(0,T]} e^{r(T-t)} dP(t)\right)\right], \tag{5.10}$$

which the payment process should follow. We remark that the pricing equation (5.10) derived for the insurer investing in a market with a risk-free and a risky asset is the same as the equation for the insurer who can only invest in a risk-free asset. If $n_0B(0)$ is the collective single premium to be paid to cover the payments P, then the insured persons do not benefit from a premium reduction which one would expect due to the higher return on a risky asset. Notice that the whole gain is taken by the insurer as the optimal value function in a market with a risk-free and a risky asset ($g < 0$) is greater than the corresponding value function in a market with a risk-free asset ($g = 0$).

We have derived the pricing equation in the case when the insurer plans to issue n policies. However, the insurer may also be interested in the marginal price of an additional n-th policy when he already has $n-1$ policies in his portfolio (see [15]). Indifference arguments yield the following pricing equation:

$$v_0(0, x + \Delta B(0))\phi_n(0, \lambda) = v_0(0, x)\phi_{n-1}(0, \lambda), \tag{5.11}$$

from which we can calculate the marginal indifference price

$$\Delta B(0) = \frac{1}{\alpha} e^{-rT} \log\Big(\frac{\phi_n(0, \lambda)}{\phi_{n-1}(0, \lambda)}\Big). \tag{5.12}$$

We point out that the price (5.12), which an individual has to pay, equals the difference of the collective premiums for portfolios consisting of n and $n-1$ policies.

5.2 Indifference pricing with respect to a random date

In this subsection we deal with the pricing principles (3.7) and (3.8). In order to obtain explicit solutions we additionally assume that the discount factor equals the risk-free rate of return, $\rho = r$.

First, let us consider the pricing equation which arises from the principles (3.7) and (3.8) in a market consisting of a risk-free asset only. Assume that the insurer issues one policy paying P at the time of death, if it occurs within T years, or at the end of the contract. The principles yield

$$-\frac{1}{\alpha} e^{-\alpha x} = \mathbb{E}\Big[-\frac{1}{\alpha} e^{-\alpha(x + B(0) - e^{-r\tau_0} P \mathbf{1}\{\tau_0 \le T\} + e^{-rT} P \mathbf{1}\{\tau_0 > T\})}\Big], \tag{5.13}$$

from which we can calculate the premium

$$B(0) = \frac{1}{\alpha} \log \mathbb{E}\Big[e^{\alpha e^{-r(\tau_0 \wedge T)} P}\Big]. \tag{5.14}$$

The derived premium is reasonable, as it corresponds to the classical exponential indifference price of the claim discounted at the time of issuing the contract. This gives motivation for investigating the indifference principles (3.7) and (3.8) in the dynamic setting. Notice that if we applied (3.8) with respect to the undiscounted wealth process, as in [6], then the price, even in our simple example, would be more complicated and would depend on the initial wealth x. This problem is not discussed in [6], as the case of $r = 0$ is only considered there.

Let us deal with the main problem. We only state our results, as the method of arriving at the solution is the same as in the previous section. We postulate the following relations:

$$w_n(t, y, \lambda) = u(y)\varphi_n(t, \lambda), \quad \forall (t, y, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty), \tag{5.15}$$

with $\varphi_0(t,\lambda) = 1$ for all $(t,\lambda) \in [0,T] \times (0,\infty)$. By substituting (5.15) into (4.16) we arrive at

$$\begin{aligned} 0 &= u(y)\frac{\partial \varphi_n}{\partial t}(t,\lambda) + \varphi_n(t,\lambda) \sup_{\pi_n \in \mathbb{R}} \left\{ \mathcal{L}^{\pi_n}_{F,\rho} u(y) \right\} + u(y)\alpha n c e^{-rt} \varphi_n(t,\lambda) \\ &\quad + u(y)\mathcal{L}_M \varphi_n(t,\lambda) + u(y) n\lambda \big(e^{\alpha b e^{-rt}} \varphi_{n-1}(t,\lambda) - \varphi_n(t,\lambda) \big), \end{aligned} \tag{5.16}$$

from which we deduce that the optimal investment strategy is the same for all n and equals

$$\tilde{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha} e^{rt}, \quad n = 1, 2, ..., n_0. \tag{5.17}$$

Notice that due to discounting, the strategy (5.17) is independent of the terminal time T. We remark that both optimal investment strategies (5.6) and (5.17) are increasing functions of time t.

The sequence of functions φ_n satisfies the system of parabolic partial differential equations

$$\begin{aligned} &0 = \frac{\partial \varphi_n}{\partial t}(t,\lambda) + \mathcal{L}_M \varphi_n(t,\lambda) + (G(\hat{\kappa}) + \alpha n c e^{-rt} - n\lambda)\varphi_n(t,\lambda) \\ &+ n\lambda e^{\alpha b e^{-rt}} \varphi_{n-1}(t,\lambda), \quad (t,\lambda) \in [0,T) \times (0,\infty), \\ &\varphi_n(T,\lambda) = e^{\alpha n B(T) e^{-rT}}. \end{aligned} \tag{5.18}$$

We conclude that $\varphi_n \in \mathcal{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathcal{C}_b([0,T] \times (0,\infty))$ for $n = 1, 2, ..., n_0$, and the probabilistic representation

$$\begin{aligned} \varphi_n(t,\lambda) &= \mathbb{E}^{t,\lambda}\Big[e^{\alpha n B(T) e^{-rT}} e^{\int_t^T (\alpha n c e^{-rs} - n\lambda(s)) ds} \\ &\quad + \int_t^T e^{\alpha b e^{-rs}} n\lambda(s) \varphi_{n-1}(s,\lambda(s)) e^{\int_t^s (\alpha c e^{-ru} - n\lambda(u)) du} ds \Big] \end{aligned} \tag{5.19}$$

holds. Similarly to Lemma 5.3 we have

$$\varphi_{n_0}(0,\lambda) = \mathbb{E}^{0,\lambda}\left[\exp\Big(G(\hat{\kappa})(\tau_0 \wedge T) + \alpha \int_{(0,T]} e^{-rt} dP(t) \Big) \right]. \tag{5.20}$$

The indifference principle (3.7) yields the pricing equation

$$n_0 B(0) = \frac{1}{\alpha} \log \mathbb{E}^{0,\lambda}\left[\exp\Big(G(\hat{\kappa})(\tau_0 \wedge T) + \alpha \int_{(0,T]} e^{-rt} dP(t) \Big) \right], \tag{5.21}$$

which takes into account the financial market in which the insurer invests. Notice that under the principle (3.7), the insured persons can benefit from a premium reduction when the insurer is allowed to invest in a risky asset ($G(\hat{\kappa}) < 0$). Moreover, the expected utility of the insurer's discounted wealth is also greater when a risky asset is available in a market compared with the case when the insurer can only invest in a risk-free asset. It turns out that both sides of the contract can derive higher gains. It is also interesting to note that the function $\kappa \mapsto G(\kappa)$ depends on the investment strategy, as $\kappa := \alpha e^{-rt}\pi$. Applying the strategy (5.17) reduces the premium by the maximum amount, while choosing $\pi = 0$ recovers the premium (5.14).

We can also give the pricing equation corresponding to the indifference principle (3.8):

$$n_0 B(0) = \frac{1}{\alpha} \log \frac{\mathbb{E}^{0,\lambda}\big[e^{G(\hat{\kappa})(\tau_0 \wedge T)} e^{\alpha \int_{(0,T]} e^{-rt} dP(t)} \big]}{\mathbb{E}^{0,\lambda}\big[e^{G(\hat{\kappa})(\tau_0 \wedge T)} \big]}, \tag{5.22}$$

which can be rewritten as

$$n_0 B(0) = \frac{1}{\alpha} \log \mathbb{E}^{\mathbb{P}^G} \left[\exp\left(\alpha \int_{(0,T]} e^{-rt} dP(t) \right) \right], \tag{5.23}$$

where the equivalent measure $\mathbb{P}^G \sim \mathbb{P}$ is defined through the Radon-Nikodym derivative

$$\frac{d\mathbb{P}^G}{d\mathbb{P}} = \frac{e^{G(\hat{\kappa})(\tau_0 \wedge T)}}{\mathbb{E}^{0,\lambda}\left[e^{G(\hat{\kappa})(\tau_0 \wedge T)}\right]}. \tag{5.24}$$

The influence of G on the price (5.22) will be investigated in the next section.

We believe that both pricing equations, derived in this section, are reasonable and economically sensible. They can be viewed as generalizations of the traditional pricing equation (5.10).

5.3 Properties of premiums

In this section we assume that $n_0 B(0)$ is the single premium which the insurer has to collect from the portfolio in order to cover the benefits $c, b, B(T)$ to be paid to the insured persons. We give some properties of $B(0)$, which is the single premium an individual has to pay. A numerical example is also considered.

The following lemma gives important properties of the derived exponential utility indifference premiums. It is interesting to note that all three premiums that we consider share the well-known properties.

Lemma 5.4. *Assume that the individual premium $B(0)$ is set according to (5.10). Then*

1. *$B^\alpha(0)$ is (strictly) increasing in α,*
2. $\lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E}\Big[\int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \le T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}\Big]$,
3. $\lim_{\alpha \to \infty} B^\alpha(0) = \mathbb{P}$*-ess sup* $\Big\{ \int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \le T\}$
   $+ B(T)e^{-rT}\mathbf{1}\{T_i > T\}\Big\}$,
4. *if $c^1 \le c^2, b^1 \le b^2, B^1(T) \le B^2(T)$ then $B^1(0) \le B^2(0)$.*

If the premium is set according to (5.21) or (5.23), then the items1, 3, 4 hold as well. For the premium (5.21) we have

2'. $\lim_{\alpha \to 0} B^\alpha(0) = -\infty$,

while for the premium (5.23) we have

2''. $\lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E}^{\mathbb{P}^G}\Big[\int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \le T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}\Big]$.

Item 2' requires an explanation. Due to the monotonicity and continuity of the mapping $\alpha \mapsto B^\alpha(0)$, arising in (5.21), there exists $\alpha^* < \infty$ such that the premium is "well-defined" for all $\alpha > \alpha^*$. The premium is meant to be well-defined if it is an economically sensible price of a cashflow, for example if it is greater than the expected value of the benefits or, simply, if it is positive. Notice that the equation (5.21) gives a reasonable price only in the case of a "high" risk averse insurer ($\alpha > \alpha^*$). We remark that in [14, 15] the price is also well-defined only for some values of the Sharpe ratio, which plays there the role of a risk aversion parameter.

We point out that the limit, for an infinitely risk averse insurer, appearing in item 3 is the cost of superhedging the financial and insurance risk. The corresponding optimal investment strategy is to invest the whole wealth in a risk-free bank account (recall the strategy (5.6)). By collecting the premium derived in item 3, the company eliminates the insurance risk, whereas investment in a bank account eliminates the financial risk.

We continue to investigate properties of the derived premiums by means of a numerical example. We assume that the mortality intensity follows the exponential Ornstein-Uhlenbeck process

$$\lambda(t) = 0,02e^{0,08t+0,1Y(t)}, \quad dY(t) = -0,2Y(t) + d\bar{W}(t), \tag{5.25}$$

whereas the price of the stock follows the exponential Variance Gamma process

$$S(t) = e^{\mu_E t + L(t)}, \quad L(t) = -0,2h(t) + 0,2W(h(t)), \tag{5.26}$$

where $h(t)$ denotes a Gamma distributed random variable with the density function

$$g_{h(t)}(y) = \frac{1}{\Gamma(t/0,003)(0,003)^{t/0,003}} y^{\frac{t}{0,003}-1} e^{-\frac{y}{0,003}}. \tag{5.27}$$

For the subordinated Brownian motion representation of Variance Gamma processes we refer the reader to Chapter 2.3 in [20]. The processes (5.25) and (5.26) appear in numerical examples in the literature, see for example [27, 19].

Let us consider a pure endowment contract with the term of $T = 10$ years and the benefit $B(T) = 100$. We investigate the impact of α, μ and n_0 on the premiums $B(0)$ calculated according to (5.10), (5.21) and (5.23). We also calculate the premium assuming deterministic mortality evolution over time $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$. In order to obtain the prices we have applied Monte-Carlo simulation.

The numerical results, presented in Tables 1-3, obviously agree with the properties stated

Table 1: The effect of α on the premiums; $\mu = 0,1, n_0 = 1$.

α	Premium for $\bar{\lambda}$	Premium (5.10)	Premium (5.21)	Premium (5.23)
0,001	49,564	50,606	-342,199	49,103
0,005	51,916	53,384	-28,686	49,964
0,01	54,295	55,801	12,283	51,585
0,02	58,134	59,295	34,780	54,516
0,04	61,399	62,792	48,472	58,255
0,05	62,807	63,901	52,514	60,426
0,06	63,154	64,465	54,945	61,418
0,1	64,085	64,932	59,832	63,753
0,15	64,876	65,767	61,980	64,558
0,2	65,293	66,183	63,283	65,112
0,6	66,659	66,648	65,892	66,441

Table 2: Reduction in the premiums due to investment in the risky asset; $\alpha = 0,05, n_0 = 1$.

μ	Reduction for the premium (5.21)	Reduction for the premium (5.23)
0,05	4,09%	3,75%
0,06	5,85%	3,91%
0,07	7,22%	4,13%
0,08	9,73%	4,29%
0,09	12,90%	4,31%
0,1	17,82%	5,44%
0,11	22,54%	6,18%
0,12	29,64%	7,89%
0,13	34,11%	8,68%
0,14	42,08%	10,09%

Table 3: The premiums per individual in the portfolio; $\alpha = 0,05, \mu = 0,1$.

n_0	Premium for $\bar{\lambda}$	Premium (5.10)
1	62,807	63,901
5	61,887	63,289
10	61,108	62,592
20	60,404	61,750
50	59,653	60,931
70	58,507	59,647
100	57,971	59,102
200	57,841	58,958

in Lemma 5.4. Notice that the premium which eliminates the financial and the insurance risk ($\alpha \to \infty$) is equal to 67.

Let us first deal with the investment strategy. For $\mu_E = 0,28$, which corresponds to $\mu = 0,1$, the coefficient $\hat{\kappa}$, which determines the optimal investment strategy, equals $\hat{\kappa} = 1,49$. If we replace the Variance Gamma process by the Brownian motion and keep the mean and variance of the driving Lévy process at the same level, then $\hat{\kappa}$ increases to 1,51. In the model with jumps it is optimal to invest a smaller amount in the risky asset. If the insurer mistakenly assumed diffusion dynamics, instead of a jump process, then he would take too much financial risk. The nice feature of the exponential indifference pricing is that the optimal strategy does not only depend on the first two moments of the distribution of the asset return but also takes into account the heavy-tailed nature of the underlying distribution.

Notice that the premium (5.10) is always greater than the premium calculated in the case of deterministic mortality. This means that the indifference principle (3.6) puts a price on the systematic mortality risk. In our simple example the systematic mortality risk premium is about $2\% - 3\%$.

It has already been stated in Section 4 that the premium can be reduced if the insurer applies the principle (3.7). It is interesting to note that the reduction is also possible under the principle (3.8). In Table 2 we give the percentages of the premium reduction. The reductions resulting from our two indifference principles (3.7) and (3.8) are significant and we believe that charging a lower premium when investing in a risky asset, is justified from the point of view of utility theory. We point out that the premium (5.21) is greater than the expected value of the liability for $\alpha > 0,045$.

We now analyze the risk structure of the premium (5.10). In the case of $\alpha = 0,05$, the premium for a single policy, set by the insurer, equals 63,901, whereas the expected liability equals 50,606 (we take the value corresponding to $\alpha = 0,001$). This gives the total risk premium in the amount of 13,295 (26%), which can be decomposed into three parts. The first part contains the risk premium, in the amount of 63,901-58,958=4,943 (9,6%), for holding one single policy instead of a portfolio with infinitely many policies. This risk can be diversified as the price (5.10) decreases when the number of policies increases. Notice that the difference between prices calculated for the deterministic and stochastic intensity does not converge to zero when increasing the number of policies but stabilizes at the level of 58,958-57,841=1,117 (2,2%). This is the second part of the total risk premium, which compensates for the systematic mortality risk, and it is charged by the insurer even after selling an arbitrarily large number of policies. The systematic mortality risk premium is rather stable (2%-3%) and does not decrease as the number of policies in a portfolio increases. This proves the well-known fact that the systematic mortality risk cannot be diversified. The third part equals 57,841-50,606=7,235 (14,2%), and it constitutes the risk premium charged by the insurer even after selling infinitely many policies and after collecting the premium for the systematic mortality risk. The third risk premium arises due to the risk aversion of the insurer, and it does not vanish as $n_0 \to \infty$. Observe, however, the decreasing pattern of the premium in the first column in Table 3.

We finish this section with a result concerning the probability of the insurer's financial insolvency. We believe that this simple estimate is worth presenting.

Lemma 5.5. *Assume that $r = 0$ and that the single premium is calculated according to (5.10). The ruin probability decreases to zero exponentially and*

$$\mathbb{P}\big(\inf_{t\in[0,T]} X^{\hat{\pi}}(t) < 0 | X(0) = x + n_0 B(0)\big) \leq e^{-\beta x + \beta n_0 \left(B^{\beta}(0) - B^{\alpha}(0)\right)}, \tag{5.28}$$

where $\beta > \alpha$ is the unique solution of the equation

$$G\Big(\beta\frac{\hat{\kappa}}{\alpha}\Big) = 0. \tag{5.29}$$

The coefficient β would be called the adjustment coefficient in the classical ruin theory and the estimate (5.28) can be viewed as a new type of Lundberg's inequality, which is very common in ruin theory (see [28]).

6 Quadratic loss function

In this section we assume that the insurer applies a quadratic loss function of the form $u(x) = -(x-\alpha)^2, \alpha > 0$. Under these preferences the insurer arranges its payment process so as to reach the desired level of wealth α, while minimizing the mean-square error. This optimization criterion is similar to the criteria applied in [4, 14, 15]. Notice that our financial market differs from those considered in [4, 14, 15].

We need a stronger assumption on the measure ν to obtain smooth solutions of the Hamilton-Jacobi-Bellman equations:

(A) $\int_{z>-1} z^4\nu(dz) < \infty$.

We do not present details of the calculations and only give some remarks. One can guess that the optimal value functions are of the form $A_n(t,\lambda)x^2 + B_n(t,\lambda)x + C_n(t,\lambda)$, where the functions A_n, B_n and C_n solve parabolic partial differential equations which are different for each optimization problem. The optimal investment strategy is given by

$$\pi_n(t) = -\frac{\mu - r}{\sigma^2 + \int_{z>-1} z^2\nu(dz)}\left(X(t-) + \frac{B_n(t,\lambda(t))}{2A_n(t,\lambda(t))}\right). \tag{6.1}$$

We point out that the optimal investment strategy takes into account the current number of survivors in the portfolio. Moreover, the optimal strategy depends not only on the currently available wealth of the insurer but also on the current value of the mortality intensity. This differs significantly from the optimal strategy for exponential utility functions.

7 Conclusions

In this paper we have investigated the problem of pricing and hedging of life insurance liabilities, in the presence of a systematic mortality risk, in a financial market with an asset driven by a Lévy process. Indifference arguments were applied in order to derive pricing equations. Explicit solutions were found for exponential and quadratic utility/loss functions. Numerical methods can be used to arrive at solutions in the case of other loss functions. It would also be interesting to investigate how the systematic mortality risk affects quantiles of the distribution of the terminal wealth.

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8 Appendix

This appendix contains the proofs of the lemmas and theorems appearing in the paper.

Proof of lemma 4.1.
Let $\pi \in \mathcal{A}$ denote an arbitrary admissible control for the problem (4.1). The following relation holds:

$$\begin{aligned}\mathbb{E}^{t,x,\lambda,n}\big[u(X^{\pi}(T))\big] &= \mathbb{E}^{t,x,\lambda,n}\big[u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\} \\ &\quad + u(X^{\pi}(T))\mathbf{1}\{\tau_{n-1} \leq T\}\big]. \end{aligned} \tag{8.1}$$

We deal with the second term. The property of conditional expectations implies that

$$\begin{aligned}&\mathbb{E}^{t,x,\lambda,n}\big[u(X^{\pi}(T))\mathbf{1}\{\tau_{n-1} \leq T\}\big] \\ &= \mathbb{E}^{t,x,\lambda,n}\big[\mathbb{E}\big[u(X^{\pi}(T))\mathbf{1}\{\tau_{n-1} \leq T\}|\mathcal{F}_{\tau_{n-1}\wedge T}\big]\big] \\ &= \mathbb{E}^{t,x,\lambda,n}\big[\mathbf{1}\{\tau_{n-1} \leq T\}\mathbb{E}\big[u(X^{\pi}(T))|\mathcal{F}_{\tau_{n-1}\wedge T}\big]\big]. \end{aligned} \tag{8.2}$$

Notice that the following inequality holds $\mathbb{P}$-a.s.:

$$\begin{aligned}
&\mathbf{1}\{\tau_{n-1} \le T\}\mathbb{E}\big[u(X^{\pi}(T)|\mathcal{F}_{\tau_{n-1}\wedge T}\big] \\
&\le \quad \mathbf{1}\{\tau_{n-1} \le T\}V_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_{n-1})\big],
\end{aligned} \tag{8.3}$$

due to the representation (3.3), the strong Markov property of the process X and the definition of the value function V_{n-1}. The equality in (8.3) is obtained by applying the optimal control on $(\tau_{n-1}, T]$ corresponding to the optimization problem with the value function V_{n-1}. □

Proof of theorem 4.1

A similar verification theorem, for $n = 1$, is proved in [29], to which the interested reader is referred for more explanations.

Fix $t \in [0, T]$ and take an arbitrary admissible control $\pi_n \in \mathcal{A}$. Let v_n denote a function which satisfies the conditions of Theorem 4.1.

We start by establishing the following relation:

$$\begin{aligned}
&\mathbb{E}^{t,x,\lambda,n}\big[v_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_n))\mathbf{1}\{\tau_{n-1} \le T\} + u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\}\big] \\
&\quad -v_n(t, x, \lambda) \\
&= \quad \mathbb{E}^{t,x,\lambda,n}\big[\{v_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_{n-1})) - v_n(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}), \lambda(\tau_{n-1}))\} \\
&\quad \times\mathbf{1}\{\tau_{n-1} \le T\}\big] \\
&\quad +\mathbb{E}^{t,x,\lambda,n}\big[v_n(\tau_{n-1} \wedge T, X_n^{\pi_n}(\tau_{n-1} \wedge T), \lambda(\tau_{n-1} \wedge T)) - v_n(t, x, \lambda)\big].
\end{aligned} \tag{8.4}$$

The first term can be rewritten as

$$\begin{aligned}
&\mathbb{E}^{t,x,\lambda,n}\big[\{v_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_{n-1})) - v_n(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}), \lambda(\tau_{n-1}))\} \\
&\quad \times\mathbf{1}\{\tau_{n-1} \le T\}\big] \\
&= \quad \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^T \mathbb{E}^{t,x,\lambda,n}\big[\big(v_{n-1}(s, X_n^{\pi_n}(s-) - b, \lambda(s)) - v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big)\big|\mathcal{F}_T^M\big] \\
&\quad \times n\lambda(s)e^{-\int_t^s n\lambda(w)dw}ds\Big] \\
&= \quad \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^T \mathbb{E}^{t,x,\lambda,n}\big[\big(v_{n-1}(s, X_n^{\pi_n}(s-) - b, \lambda(s)) - v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big) \\
&\quad \times n\lambda(s)\mathbf{1}\{\tau_{n-1} \ge s\}|\mathcal{F}_T^M\big]ds\Big] \\
&= \quad \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^T n\lambda(s)\big(v_{n-1}(s, X_n^{\pi_n}(s-) - b, \lambda(s)) - v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big) \\
&\quad \times\mathbf{1}\{\tau_{n-1} \ge s\}ds\Big]
\end{aligned} \tag{8.5}$$

where we have applied: the property that $X_n^{\pi_n}(\tau_{n-1}-) = X_n^{\pi_n}(\tau_{n-1})$ holds $\mathbb{P}$-a.s., the property of conditional expectations, the distribution of τ_{n-1} given the filtration $\mathcal{F}_T^M$, the independence of the random variables $\mathbf{1}\{\tau_{n-1} \ge s\}$, $X_n^{\pi_n}(s-)$ and the Fubini theorem.

In order to handle the second term in (8.4) we introduce the localizing sequence $t_m = \inf\{s \in (t, T]; |X_n^{\pi_n}(s) - x| + |\lambda(s) - \lambda| + |\pi_n(s)| > m\}$. By choosing an arbitrary $\varepsilon \in (0, T - t)$ and applying the Itô formula for semimartingales (see Theorem 4.4.7 in [17]) we arrive at

$$\begin{aligned}
&\mathbb{E}^{t,x,\lambda,n}\big[v_n(\tau_{n-1}\wedge t_m\wedge(T-\varepsilon), X_n^{\pi_n}(\tau_{n-1}\wedge t_m\wedge(T-\varepsilon)), \lambda(\tau_{n-1}\wedge t_m\wedge(T-\varepsilon)))\big]\\
&\quad -v_n(t,x,\lambda)\\
&= \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^{T-\varepsilon}\Big(\frac{\partial v_n}{\partial t}(s, X_n^{\pi_n}(s-),\lambda(s)) + \mathcal{L}_F^{\pi_n(s)} v(s, X_n^{\pi_n}(s-),\lambda(s))\\
&\quad -\frac{\partial v_n}{\partial x}(s, X_n^{\pi_n}(s),\lambda(s))nc + \mathcal{L}_M v_n(s, X_n^{\pi_n}(s-),\lambda(s))\Big)\mathbf{1}\{\tau_{n-1}\geq s, t_m\geq s\}ds\Big].
\end{aligned} \tag{8.6}$$

We point out that the expected values of the stochastic integrals with respect to the Brownian motions and the Poisson random measure are equal to zero due to the condition (4.8) and the localizing sequence.

The next steps are rather standard (see for example Theorem 3.1 in [24]). In order to prove (4.11), first combine (8.5) with (8.6), then apply inequality (4.6) and let $m\to\infty$ and $\varepsilon\to 0$. In order to arrive at (4.14) apply the strategy (4.12). □

We point out that, due to our localizing sequence, we can get rid of some of the conditions stated in Theorem 3.1 in [24], which would be difficult to check when verifying the optimality of the solution.

Proof of lemma 5.2

We follow the proof of Proposition 2.3 in [22]. Choose $\varepsilon>0$ and consider the partial differential equation (5.7) on the time interval $[0,T-\varepsilon]$. The function

$$\begin{aligned}
\phi_n(t,\lambda) &= \mathbb{E}^{t,\lambda}\Big[\phi_n(T-\varepsilon,\lambda(T-\varepsilon))e^{\int_t^{T-\varepsilon}(\alpha f(s)nc-n\lambda(s))ds}\\
&\quad+\int_t^{T-\varepsilon} e^{\alpha f(s)b} n\lambda(s)\phi_{n-1}(s,\lambda(s))e^{\int_t^s(\alpha f(u)nc-n\lambda(u))du}ds\Big]
\end{aligned} \tag{8.7}$$

is the unique solution of the equation (5.7) on the time interval $[0,T-\varepsilon]\times(0,\infty)$. We have $\phi_n\in\mathcal{C}^{1,2}([0,T-\varepsilon)\times(0,\infty))\times\mathcal{C}_b([0,T-\varepsilon]\times(0,\infty))$, provided that we show that the conditions from Theorem 1 in [21] are satisfied. By the assumption, the function ϕ_{n-1} is uniformly Hölder continuous on compact subsets of $[0,T-\varepsilon]\times\bar{D}_n$, so we only have to show that the mapping $(t,\lambda)\mapsto\phi_n(t,\lambda)$, defined in (5.8), is continuous.

Notice that the mapping

$$\begin{aligned}
&(t,\lambda)\mapsto e^{\alpha nB(T)}e^{\int_t^T(\alpha f(s)nc-n\lambda^{t,\lambda}(s))ds}\\
&+\textstyle\int_t^T e^{\alpha f(s)b}n\lambda^{t,\lambda}(s)\phi_{n-1}(s,\lambda^{t,\lambda}(s))e^{\int_t^s(\alpha f(u)nc-n\lambda^{t,\lambda}(u))du}ds
\end{aligned} \tag{8.8}$$

is $\mathbb{P}$-a.s. uniformly bounded in (t,λ), with the bound depending on the bound of ϕ_{n-1}. The continuity of the mapping $(t,\lambda)\mapsto\phi_n(t,\lambda)$ now follows from the Lebesgue dominated convergence theorem and the continuity of the mapping $(t,\lambda)\mapsto\lambda^{t,\lambda}(s)$.

As ε is arbitrary, the existence of the solution ϕ_n, its probabilistic representation (5.8) and its smoothness on $[0,T]\times(0,\infty)$ follows. □

Proof of lemma 5.3

Notice that

$$\begin{aligned}\phi_n(t,\lambda) &= \mathbb{E}^{t,\lambda,n}\Big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\} \\ &\quad +e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu+\alpha f(\tau_{n-1})b}\phi_{n-1}(\tau_{n-1},\lambda(\tau_{n-1}))\mathbf{1}\{\tau_{n-1}\le T\}\Big], \end{aligned} \tag{8.9}$$

holds by the property of conditional expectations and the distribution of the random variable τ_{n-1} given the filtration $\mathcal{F}_T^M$. By applying again the property of conditional expectations and the strong Markov property we can arrive at

$$\begin{aligned}&\phi_n(t,\lambda)\\ &= \mathbb{E}^{t,\lambda,n}\Big[\mathbb{E}\Big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\}\\ &\quad +e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu+\alpha f(\tau_{n-1})b}\phi_{n-1}(\tau_{n-1},\lambda(\tau_{n-1}))\mathbf{1}\{\tau_{n-1}\le T\}\big|\mathcal{F}_{\tau_{n-1}\wedge T}\Big]\Big]\\ &= \mathbb{E}^{t,\lambda,n}\Big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\}+e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu+\alpha f(\tau_{n-1})b}\\ &\quad \times\mathbb{E}\Big[e^{\int_{\tau_{n-1}}^T \alpha f(s)(n-1)cds+\alpha(n-1)B(T)}\mathbf{1}\{\tau_{n-2}>T\}+e^{\int_{\tau_{n-1}}^{\tau_{n-2}} \alpha f(u)(n-1)cdu+\alpha f(\tau_{n-2})b}\\ &\quad \times\phi_{n-2}(\tau_{n-2},\lambda(\tau_{n-2}))\mathbf{1}\{\tau_{n-2}\le T\}\big|\mathcal{F}_{\tau_{n-1}\wedge T}\Big]\mathbf{1}\{\tau_{n-1}\le T\}\Big]\\ &= \mathbb{E}^{t,\lambda,n}\Big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\}\\ &\quad +e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu+\alpha f(\tau_{n-1})b+\int_{\tau_{n-1}}^T \alpha f(s)(n-1)cds+\alpha(n-1)B(T)}\mathbf{1}\{\tau_{n-1}\le T,\tau_{n-2}>T\}\\ &\quad +e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu+\alpha f(\tau_{n-1})b+\int_{\tau_{n-1}}^{\tau_{n-2}} \alpha f(u)(n-1)cdu+\alpha f(\tau_{n-2})b}\phi_{n-2}(\tau_{n-2},\lambda(\tau_{n-2}))\mathbf{1}\{\tau_{n-2}\le T\}\Big]\end{aligned} \tag{8.10}$$

By continuing the calculations we arrive at the representation (5.9) for $n=n_0$ and $t=0$. □

Proof of lemma 5.4

Properties 1-4 of the premium (5.10) are well-known (see for example Theorem 3.1.1 in [28]). These properties can also be extended for the premium (5.23). As the expectation in (5.23) is taken under the measure $\mathbb{P}^G$, the expected value, arising in the limit $\alpha\to 0$, has also to be taken under the measure $\mathbb{P}^G$. Property 3 remains unchanged due to the equivalence of the measures.

We can now deal with the premium (5.21). Properties 2 and 4 clearly hold true. To prove property 1 take $\alpha_1\le\alpha_2$ and apply Jensen's inequality to arrive at

$$\begin{aligned}&\Big(\mathbb{E}\Big[e^{G(\hat\kappa)(\tau_0\wedge T)+\alpha_1\int_0^T e^{-rt}dP(t)}\Big]\Big)^{\frac{\alpha_2}{\alpha_1}}\\ &\le \mathbb{E}\Big[e^{\frac{\alpha_2}{\alpha_1}G(\hat\kappa)(\tau_0\wedge T)+\alpha_2\int_0^T e^{-rt}dP(t)}\Big]\le\mathbb{E}\Big[e^{G(\hat\kappa)(\tau_0\wedge T)+\alpha_2\int_0^T e^{-rt}dP(t)}\Big],\end{aligned} \tag{8.11}$$

where the second inequality follows from the negativity of $G(\hat\kappa)$. To prove property 3 notice that

$$\begin{aligned}&\frac{G(\hat\kappa)T}{\alpha}+\frac{1}{\alpha}\log\mathbb{E}\Big[e^{\alpha\int_0^T e^{-rt}dP(t)}\Big]\\ &\le \frac{1}{\alpha}\log\mathbb{E}\Big[e^{G(\hat\kappa)(\tau_0\wedge T)+\alpha\int_0^T e^{-rt}dP(t)}\Big]\le\frac{1}{\alpha}\log\mathbb{E}\Big[e^{\alpha\int_0^T e^{-rt}dP(t)}\Big].\end{aligned} \tag{8.12}$$

The proof is completed by taking the limit $\alpha\to\infty$. □

Proof of lemma 5.5

Notice that

$$X_0^{x,\hat{\pi}_0}(t) = x + \frac{\hat{\kappa}}{\alpha}\Big(\mu t + \sigma W(t) + \int_0^T \int_{z>-1} z\tilde{M}(dz \times dt)\Big) \tag{8.13}$$

is a Lévy process. It follows from Chapter 5.4.5 in [17] that the process $\exp\big(-\beta X_0^{x,\hat{\pi}_0}(t) - tG(\beta\frac{\hat{\kappa}}{\alpha})\big)$ is a martingale. As $X^{x,\hat{\pi}}(t) = X_0^{x,\hat{\pi}_0}(t) + n_0 B(0) - \int_0^t dP(s)$ holds $\mathbb{P}$-a.s. and the payment process is increasing, we conclude that the process $\exp\big(-\beta X^{x,\hat{\pi}}(t) - tG(\beta\frac{\hat{\kappa}}{\alpha})\big)$ is a submartingale.

Choose β such that $G(\beta\frac{\hat{\kappa}}{\alpha}) = 0$. The existence of $\beta > \alpha$ follows from the properties of the function G. By applying the submartingale inequality (see Theorem 10.2.1 in [28]) we arrive at

$$\begin{aligned}
&\mathbb{P}\big(\inf_{t\in[0,T]} X^{x,\hat{\pi}}(t) \le 0\big) = \mathbb{P}\big(\sup_{t\in[0,T]} e^{-\beta X^{x,\hat{\pi}}(t)} \ge 1\big)\\
\le\ & \mathbb{E}\Big[e^{-\beta X^{x,\hat{\pi}}(T)}\Big] = \mathbb{E}\Big[e^{-\beta(X_0^{x,\hat{\pi}_0}(T) + n_0 B(0) - \int_0^T dP(t))}\Big]\\
=\ & \mathbb{E}\Big[e^{-\beta X_0^{x,\hat{\pi}_0}(T)}\Big] e^{-\beta n_0 B(0)} \mathbb{E}\Big[e^{\beta \int_0^T dP(t)}\Big] = e^{-\beta x - \beta n_0 B^{\alpha}(0) + n_0 \beta B^{\beta}(0)},
\end{aligned} \tag{8.14}$$

which completes the proof. □

The proofs that the investment strategy and our candidate value function satisfy the verification theorem are left to the reader. The details of the calculations can be obtained from the author upon request.

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