An optimal investment strategy for a stream of liabilities generated by a step process in a financial market driven by a Lévy process

Łukasz Delong*

Institute of Econometrics, Division of Probabilistic Methods,

Warsaw School of Economics

Niepodległości 162, 02-554 Warsaw, Poland

lukasz.delong@sgh.waw.pl

Abstract

In this paper we investigate an asset-liability management problem for a stream of liabilities written on liquid traded assets and non-traded sources of risk. We assume that the financial market consists of a risk-free asset and a risky asset which follows a geometric Lévy process. The non-tradable factor (insurance risk or default risk) is driven by a step process with a stochastic intensity. Our framework allows us to consider financial risk, systematic and unsystematic insurance loss risk (including longevity risk), together with possible dependencies between them. An optimal investment strategy is derived by solving a quadratic optimization problem with a terminal objective and a running cost penalizing deviations of the insurer's wealth from a specified profit-solvency target. Techniques of backward stochastic differential equations and the weak property of predictable representation are applied to obtain the optimal asset allocation.

Keywords: backward stochastic differential equation, weak property of predictable representation, quadratic optimization, equity-linked payment process, unsystematic and systematic insurance risk.
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1 Introduction

We investigate an asset-liability management problem for a stream of liabilities written on liquid traded assets and non-traded sources of risk. In insurance business values of liabilities and timing of claims depend not only on non-market risk factors, like mortality or claim propensity, but are very often related to the financial market. Complicated structures of modern insurance products and non-trivial risks involved in running an insurance company call for a sound asset-liability management and derivation of hedging strategies.

In this paper we deal with a quadratic optimization or a quadratic hedging which is defined as the choice of an investment portfolio found by minimizing, in the mean square sense, an error of not fulfilling the contractual obligations. We consider a generalized penalty function, not only with a terminal objective, but also with a running cost. Such an optimization criterion has already been applied in insurance and finance in the context of asset allocations: see Detemple, Rindisbacher (2008), Gerrard et al. (2004), Kohlmann, Peisl (2000). We believe that the inclusion of the running costs is very reasonable as it stabilizes deviations of the wealth process over the whole investment period and forces the wealth process to meet the desired targets. A trajectory of the wealth process is subject to a penalty each time it does not fulfill the target. We can apply arbitrary targets, but the particular target we propose has a clear economic interpretation. It combines the insurer's preferences concerning the future expected profit and requirements on the capital, which has to be held over the contract duration, to cover the liability reserve. The result of our optimization problem, based on minimizing deviations with respect to the proposed target, is the asset allocation strategy generating wealth values which are as close as possible, in the mean square sense, to the levels under which the company earns the desired profit and fulfills the statutory reserve requirements.

We state and solve our quadratic optimization problem under the real-world measure/objective measure. In this respect our optimization problem is close to mean-variance Markowitz portfolio selection under which extreme deviations, profits and losses are penalized under the real-world expectations. Our formulation is fundamentally different from the quadratic hedging problem/risk minimization under a martingale measure (see Ankirchner, Imkeller (2008), Dahl et al. (2007), Dahl, Møller (2006), Møller (2001)) or local risk-minimization (see Schweizer (2008), Vandaele, Vanmaele (2008)) which require specification of a martingale measure (more or less arbitrary) or finding a proper martingale measure (which might be very difficult in some cases). As far as quadratic optimization under the real-world measure is concerned, there is a large literature on extensions of the Markowitz problem of minimizing variance given expected value or minimizing the mean square-error, and we just cite the recent papers of Xie (2009) and Xie et al. (2008), where insurance liabilities follow, respectively, a Brownian motion and a Markovian regime-switching Brownian motion, and where financial models based on a geometric Brownian motion and a Markovian regime-switching Brownian motion are investigated. See as well Delong et al. (2008) where mean-variance hedging of an annuity under stochastic mortality is investigated in a financial model based on an exponential Lévy process.

We consider a financial market consisting of a risk-free asset, with deterministic return, and a risky asset whose price dynamics is described by an exponential Lévy process. It is well-known, see Cont, Tankov (2004), Kyprianou et al. (2005), that Lévy processes, unlike Brownian motions, can easily reproduce heavy tails, skewness and other distributional properties of assets' returns, and, what is very important, can generate discontinuities, sudden jumps, in price dynamics and capture price movements in a much better way. Nowadays, it is widely accepted that in the celebrated Black-Scholes model one should replace a Gaussian noise by a more general Lévy noise.

The liabilities which we investigate in this paper are related to the financial market and also depend, as is always the case for insurance products, on a non-tradeable factor. The payment process considered is similar to the payment processes from Dahl et al. (2007), Dahl, Møller (2006), Møller (2001), but simultaneously takes into account equity-linked liabilities and unsystematic and systematic insurance risk. To the best of our knowledge, this is the first paper in the actuarial literature where an asset allocation strategy or a hedging strategy is derived for a stream of claims involving financial risk and unsystematic and systematic insurance risk. In our opinion integration of these three risk factors is very important for risk management. By unsystematic insurance risk we understand randomness of insurance claims (number of accidents or number of deaths) compared to expectation, and by systematic insurance risk we mean unpredictable changes in the underlying claim intensity (clients' claim frequency or population mortality intensity). The most important example of systematic insurance risk is the longevity risk. We recall that in Møller (2001) risk minimization hedging under a martingale measure and mean-variance optimization are considered for a stream of equity-linked liabilities, only under unsystematic insurance risk, in a continuous Black-Scholes financial model, whereas in Dahl et al. (2007) and Dahl, Møller (2006) risk minimization hedging under a martingale measure is considered only for traditional liabilities (without a financial component), under unsystematic and systematic mortality risk, in a financial model consisting of a bank account with a risk-free return driven by a CIR process, a bond and a mortality derivative. Finally in Vandaele, Vanmaele (2008) local risk minimization hedging is investigated for an equity-linked pure endowment and term insurance, only under unsystematic mortality risk, in a financial market based on an exponential Lévy process.

In this paper we deal with a loss process consisting of continuously paid benefits, a terminal benefit and randomly occurring benefits. Our payment process is based on a step process (or a random measure) with a stochastic intensity and a random transition probability, with possible dependencies between tradeable and non-tradeable factors, and goes far beyond the processes considered in Dahl et al. (2007), Dahl, Møller (2006), Møller (2001) or Vandaele, Vanmaele (2008). It can also include a regime switching dynamics of claims in the spirit of Xie (2009). We believe that our loss process should cover all interesting and most common real-life payment schemes. In taking a step process (a random measure) as claims' driving process we follow closely Becherer (2006), where exponential utility optimization with a terminal liability is investigated. We also mention another related paper by Ankirchner, Imkeller (2008), where an integrated financial and insurance model with correlated tradeable and non-tradeable risk factors driven by Brownian motions and a Lévy process is considered, which in some aspects is more general than ours but in many aspects less general, and a quadratic hedging problem with a terminal claim, stated under a martingale measure, is investigated. Finally, let us notice that our payment process can also describe claims from defaultable securities and one can apply our optimization problem to hedge defaultable instruments (see for example Bielecki, Jeanblanc (2005)).

In order to solve the stated optimization problem we apply techniques of backward stochastic differential equations and the property of weak predictable representation of local martingales. We point out that the approach in terms of Hamilton-Jacobi-Bellman equations is not applicable. BSDEs provide a powerful tool which allows handling many state variables and dependencies between them. As concerns mathematics, we follow Becherer (2006), Lim (2005), Øksendal, Hu (2008) and Møller (2001). The idea of solving our optimization problem is known but technically differs from the above papers and introduces new difficulties. We point out that, in contrast to our paper, Becherer (2006) and Øksendal, Hu (2008) do not provide any explicit solutions of backward stochastic differential equations. We first characterize the optimal investment strategy in terms of a solution of a backward stochastic differential equation driven by Brownian motions and random measures. Next, for two specific, but still very general, equity-linked payment processes with independent unsystematic and systematic insurance risk, we derive an explicit characterization. Finally, we mention that in Ankirchner, Imkeller (2008) a quadratic hedging strategy is characterized, not in the terms of a solution of the corresponding backward stochastic differential equation, but by Malliavin derivatives.

This paper is structured as follows. In Section 2 we introduce the financial model and the stream of liabilities. The optimization problem is stated in Section 3 and the solution is derived in Section 4. In Section 5 we consider two special cases of optimization under a martingale measure and the case of upward bounded jumps in the risky asset price dynamics. An explicit strategy for equity-linked life insurance and non-life insurance payment processes with independent unsystematic and systematic insurance loss risk is derived in Section 6, which also contains a numerical example. The appendix contains some additional results and proofs omitted in the main text.

2 The model

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ and a finite time horizon $T < \infty$. We assume that $\mathbb{F}$ satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). The measure $\mathbb{P}$ is the real-world, objective probability measure and, if not specified, all expected values are taken with respect to $\mathbb{P}$. By $\mathcal{B}(A)$ we denote the Borel subsets of A, and by K a constant whose value may change from line to line.

In the following subsections we introduce the financial market and the payment process generated by the stream of liabilities.

2.1 The financial market

The financial market we consider consists of two tradeable instruments. The price of a risk-free asset $S_0 := (S_0(t), 0 \le t \le T)$ is described by the ordinary differential equation

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad S_0(0) = 1, \tag{2.1}$$

and the dynamics of the risky asset's price $S := (S(t), 0 \le t \le T)$ is given by the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + dL(t), \quad S(0) = s_0 > 0, \tag{2.2}$$

where r denotes the risk-free rate of interest, μ denotes the expected return on the risky asset and $L := (L(t), 0 \le t \le T)$ is a zero-mean Lévy process. We assume that L is $\mathbb{F}$-adapted and we deal with its càdlàg modification, i.e. with a process whose sample paths are continuous on the right and have limits on the left, see Theorem 2.1.7 in Applebaum (2004). In recent years, the model (2.1)-(2.2) has become standard in financial mathematics.

Recall that a Lévy process is a process with independent and stationary increments, which is stochastically continuous, see Chapter 1.3 in Applebaum (2004).

The zero-mean process L satisfies the Lévy-Itô decomposition, see Theorem 2.4.1 in Applebaum (2004) and Theorem 11.45 in He et al. (1992),

$$L(t) = \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}-\{0\}} z\big(M(ds,dz) - \nu(dz)ds\big), \quad 0 \le t \le T,$$

where $W := (W(t), 0 \le t \le T)$ is a Brownian motion and M a random measure on $\Omega \times \mathcal{B}((0,T]) \times \mathcal{B}((\mathbb{R}-\{0\}))$ which is independent of W. The random measure M

$$M(dt,dz) = \sum_{s\in(0,T]} \mathbf{1}_{(s,\triangle L(s))}(dt,dz)\mathbf{1}_{\{\triangle L(s)\neq 0\}}(s),$$

with $\triangle L(s)$ denoting the increment $L(s) - L(s-)$, counts the number of jumps of a given size. The measure ν, defined on $\mathcal{B}(\mathbb{R}-\{0\})$, is σ-finite, and $\lambda \otimes \nu$, where λ denotes Lebesgue measure, is the compensator (or the dual predictable projection) of the measure M. The compensated random measure is defined by

$$\tilde{M}(dt,dz) = M(dt,dz) - \nu(dz)dt.$$

We set $M(\{0\}, \mathbb{R}-\{0\}) = M((0,T],\{0\}) = \nu(\{0\}) = 0$.

We make the following assumptions concerning the coefficients and the compensator:

(A1) r, μ, σ are non-negative constants and $0 \le r \le \mu$,

(A2) the measure ν is defined on $(-1,\infty)$ and satisfies $\int_{z>-1} z^2\nu(dz) < \infty$.

Clearly, $\sigma^2 + \int_{z>-1} z^2\nu(dz) > 0$ must hold, otherwise S is risk-free. The condition **(A1)** needs no justification as it is obvious from economic point of view. We remark that the relation $\mu = r = 0$ corresponds to the case when S is already a martingale under $\mathbb{P}$. Condition **(A2)** guarantees that the jumps of the Lévy process L are bounded from below by -1 and ensures that the price process S remains strictly positive, see Proposition 5.1 in Applebaum (2004). Moreover, the integrability of ν implies that $(\int_0^t \int_{\mathbb{R}} z\tilde{M}(ds,dz))_{0\le t\le T}$ is a square integrable martingale, see Theorem 4.2.3 in Applebaum (2004). We point out that we always integrate on $\mathbb{R}$ and recall that on $(-\infty,-1]$ the integral is zero.

Under **(A2)** the stochastic differential equation (2.2) has the unique, positive and a.s. finite solution, given explicitly by

$$\begin{aligned} S(t) &= \exp\Big\{\Big(\mu - \frac{1}{2}\sigma^2 + \int_{z>-1} (\log(1+z) - z)\nu(dz)\Big)t \\ &\quad +\sigma W(t) + \int_0^t \int_{z>-1} \log(1+z)\tilde{M}(ds,dz)\Big\} \\ &= \exp\Big\{\mu_E t + \sigma W(t) + \int_0^t \int_{\mathbb{R}} z\tilde{M}_E(ds,dz)\Big\} = e^{L_E(t)}, \end{aligned} \tag{2.3}$$

which is the exponential Lévy process $L_E := (L_E(t), 0 \leq t \leq T)$ with the measure $\nu_E(A) = \nu(\{z : \log(1+z) \in A\})$, see Propositions 8.21 and 8.22 in Cont, Tankov (2004). There is one-to-one correspondence between the measures and the models (2.2) and (2.3). From a practical point of view it is more convenient to start with the exponential dynamics (2.3) and define $\tilde{M}_E$ as the measure corresponding to a Lévy process L_E with unbounded jumps. Some popular choices of L_E include: Merton model, Kou model, variance gamma process, normal inverse gaussian process, generalized hyperbolic distributions, tempered stable processes. All these processes have been applied with success in many different areas of financial modelling: see Cont, Tankov (2004) and Kyprianou et al. (2005).

2.2 The stream of liabilities

We deal with a very general stream of insurance liabilities with possible interactions with the financial market. In our opinion the loss process presented in this section covers all interesting payment schemes that may occur in insurance, reinsurance and pensions.

First, we define $Y := (Y(t), 0 \leq t \leq T)$ to be an $\mathbb{F}$-adapted step process. We remark that by a step process we mean a process with càdlàg step sample paths that have at most a finite number of jumps in every finite time interval, see Definition 11.48 in He et al. (1992). With the process Y we can associate the random measure

$$N(dt, dy) = \sum_{s \in (0,T]} \mathbf{1}_{(s, \triangle Y(s))}(dt, dy) \mathbf{1}_{\{\triangle Y(s) \neq 0\}}(s), \tag{2.4}$$

defined on $\Omega \times \mathcal{B}((0,T]) \times \mathcal{B}(\mathbb{R} - \{0\})$, with its unique compensator ϑ (the predictable dual projection of N), see Theorem 11.15 in He et al. (1992). We assume that

(A3) the compensator ϑ, defined on $\Omega \times \mathcal{B}((0,T]) \times \mathcal{B}(\mathbb{R} - \{0\})$, is of the form

$$\vartheta(dt, dy) = Q(t, dy)\xi(t)dt,$$

and satisfies

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}-\{0\}} Q(t, dy)\xi(t)dt\Big] < \infty,$$

where $\xi : \Omega \times [0,T] \to [0, \infty)$ is a predictable process; for a fixed $(\omega, t) \in \Omega \times (0,T]$, $Q(\omega, t, .)$ is a probability measure on $\mathbb{R} - \{0\}$, and, for a fixed $A \in \mathcal{B}(\mathbb{R} - \{0\})$, $Q(., ., A) : \Omega \times [0,T] \to [0,1]$ is a predictable process.

We set $N(\{0\}, \mathbb{R} - \{0\}) = N((0,T], \{0\}) = \vartheta((0,T], \{0\}) = 0$. By predictability of a mapping $\Omega \times [0,T] \to \mathbb{R}$ we mean measurability with respect to the σ-algebra $\mathcal{P}$ generated by all left-continuous $\mathbb{F}$-adapted processes. The assumption **(A3)** has an intuitive interpretation, where ξ stands for the intensity of jumps of the process Y and Q gives the distribution of a jump's height given that a jump occurs.

We also consider a background driving factor represented by the process $\Lambda := (\lambda(t), 0 \leq t \leq T)$ with the dynamics given by the stochastic differential equation

$$d\lambda(t) = a(t, \lambda(t))dt + b(t, \lambda(t))dB(t), \quad \lambda(0) = \lambda_0, \tag{2.5}$$

where $B := (B(t), 0 \leq t \leq T)$ denotes an $\mathbb{F}$-adapted Brownian motion and we assume that

(A4) there exists a unique, $\mathbb{F}$-adapted, continuous solution Λ of (2.5) on $[0, T]$,

(A5) the processes W and B are independent.

The process Λ affects the intensity ξ and the distribution Q. Notice that we allow the drift a and the volatility b in (2.5) to depend on the Lévy process L or the step process Y. In order to clarify the role of Λ as a background driving factor, think about a portfolio of risks and let Λ describe a random mortality intensity of a single life or a random claim intensity of an individual policy. Then ξ provides the intensity for a portfolio which can depend on other factors not captured by Λ, for example it can depend on the current number of risks in a portfolio modelled by Y.

In this paper we investigate the stream of liabilities with the corresponding payment process $P := (P(t), 0 \leq t \leq T)$ described by

$$P(t) = \int_0^t H(s)ds + \int_0^t \int_{\mathbb{R}} G(s, y)N(ds, dy) + F\mathbf{1}_{t=T}. \tag{2.6}$$

The process (2.6) may model payments arising from various integrated insurance and financial products. The loss process P contains payments H which occur continuously during the term of the contract; it contains claims G which occur at random times and liabilities F which are settled at the end of the contract. Based on the representation (2.4), we can conclude that

$$\int_0^t \int_{\mathbb{R}} G(s, y)N(ds, dy) = \sum_{s \in (0,t]} G(s, \triangle Y(s))\mathbf{1}_{\{\triangle Y(s) \neq 0\}}(s), \quad 0 \leq t \leq T, \mathbb{P}-\text{a.s.},$$

and we can realize that the stochastic integral with respect to the random measure N models the claims occurring at times when the step process Y jumps. Notice that H, G and F can depend (also in a pathwise sense) on different sources of uncertainty captured by the filtration $\mathbb{F}$; they can depend on variables related to the performance of the financial market driven by L or on variables related to (Y, Λ). Moreover, ξ and Q are allowed to depend on (L, Y, Λ). Using this construction we can consider interesting and non-trivial dependency structures between tradeable and non-tradeable risk factors. We do not assume that H, G, F are nonnegative and we are able to take into account possible inflows (future premiums) as well. Moreover, the inclusion of Λ allows us to deal with a background driving risk factor. A prime example of risk induced by Λ is systematic insurance loss risk which cannot be diversified.

We assume that

(A6) $H : \Omega \times [0,T] \to \mathbb{R}$ is an $\mathbb{F}$-adapted, product measurable process and $|H|$ is an a.s. finite process satisfying

$$\mathbb{E}\big[\int_0^T \big|H(t)\big|^2 dt\big] < \infty,$$

(A7) $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a predictable process satisfying

$$\mathbb{E}\big[\int_0^T \int_{\mathbb{R}} \big|G(t,y)\big|^2 \xi(t) Q(t,dy)dt\big] < \infty,$$

$$\mathbb{E}\big[\int_0^T \int_{\mathbb{R}} \big|G(t,y)\big|^2 |\xi(t)|^2 Q(t,dy)dt\big] < \infty,$$

(A8) $F : \Omega \to \mathbb{R}$ is an $\mathcal{F}_T$-measurable random variable satisfying

$$\mathbb{E}\big[|F|^2\big] < \infty,$$

(A9) the Lévy process L and the process Y never jump together, i.e. the quadratic variation $[L, Y]_t$ is 0 for $0 \le t \le T$.

Here, by predictability of a mapping $\Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ we mean measurability with respect to the product σ-field $\mathcal{P} \times \mathcal{B}(\mathbb{R})$. The requirements of square integrability are standard when dealing with quadratic optimization. The assumptions **(A6)**-**(A8)** imply that that the process P is square integrable:

$$\begin{aligned}
\mathbb{E}\big[\sup_{t\in[0,T]} |P(t)|^2\big] &\le K\Big(\mathbb{E}\big[\int_0^T \big|H(t)\big|^2 dt\big] + \mathbb{E}\big[\sup_{t\in[0,T]} \big|\int_0^t \int_{\mathbb{R}} G(s,y)\tilde{N}(ds,dy)\big|^2\big] \\
&\quad + \mathbb{E}\big[\big|\int_0^T \int_{\mathbb{R}} \big|G(t,y)\big|\xi(t)Q(t,dy)dt\big|^2\big] + \mathbb{E}\big[|F|^2\big]\Big) \\
&\le K\Big(\mathbb{E}\big[\int_0^T \big|H(t)\big|^2 dt\big] + \mathbb{E}\big[\int_0^T \int_{\mathbb{R}} \big|G(t,y)\big|^2 \xi(t)Q(t,dy)dt\big] \\
&\quad + \mathbb{E}\big[\int_0^T \int_{\mathbb{R}} |G(t,y)\xi(t)|^2 Q(t,dy)dt\big] + \mathbb{E}\big[|F|^2\big]\Big) < \infty, \qquad (2.7)
\end{aligned}$$

by Cauchy-Schwarz' inequality and Doob's martingale inequality. In particular, see Definition 11.16 in He et al. (1992), the process $(\int_0^t \int_{\mathbb{R}} G(s,y)\tilde{N}(ds,dy))_{0\le t\le T}$ is a square integrable martingale with the quadratic variation $[.,.]_t$ process satisfying

$$\mathbb{E}\Big\{\big[\int_0^{\cdot}\int_{\mathbb{R}} G(s,y)\tilde{N}(ds,dy), \int_0^{\cdot}\int_{\mathbb{R}} G(s,y)\tilde{N}(ds,dy)\big]_t\Big\}$$
$$= \mathbb{E}\big[\int_0^t \int_{\mathbb{R}} \big|G(t,y)\big|^2 \xi(t) Q(t,dy)dt\big], \quad 0 \le t \le T.$$

The assumption **(A9)** is satisfied in many situations. We remark that one can always construct dependent processes L and Y for which $[L,Y]=0$, see Example 4.5 in Becherer (2006).

In insurance and credit risk applications the assumption that the claims are generated by the step process should be sufficient. However, we can follow the theoretical model from Becherer (2006). We can start with the definition of an abstract random measure N, see Definition 11.2 in He et al. (1992), and assume that the payment process (2.6) is driven by the (compensated) random measure. With this construction, we can go beyond the class of step processes.

We investigate in more detail two special cases of the payment process (2.6) which seem to be the most common in life and non-life insurance business. In Section 6 we derive explicit hedging strategies for them.

Let us comment on the notation. By **A** we denote general assumptions concerning the financial market, the stream of liabilities and the optimization problem, by **B** and **C** we denote specific assumptions concerning the stream of liabilities in life insurance or the stream of liabilities in non-life insurance, and by **D** we denote specific assumptions which hold for both streams of liabilities in life and non-life insurance.

2.2.1 The first special case: the stream of liabilities in life insurance

In life insurance we usually deal with a portfolio consisting of identical policies issued to a group of n persons with the term of T years. Let Y count the number of deaths in a portfolio and let Λ denote the underlying stochastic mortality intensity of individuals in the portfolio. Following Dahl et al. (2007), Dahl, Møller (2006), Møller (2001), we investigate the payment process P given by

$$\begin{aligned} P(t) &= \int_0^t (n - Y(s))h(s,S(s))ds + \int_0^t \int_{\mathbb{R}} g(s,S(s-))N(ds,dy) \\ &\quad + (n-Y(T))f(T,S(T))\mathbf{1}_{t=T}, \quad 0 \le t \le T, \end{aligned} \tag{2.8}$$

with the step process Y under the characteristics

$$\begin{aligned} \xi(t) &= (n - Y(t-))\lambda(t), \quad 0 \le t \le T, \\ Q(t,\{1\}) &= 1, \quad 0 \le t \le T, \end{aligned} \tag{2.9}$$

and we assume that

(B1) the mortality intensity is independent of the financial market and the number of deaths in the portfolio, i.e. Λ is independent of L, Y,

(B2) the future lifetimes $\tau_1, ..., \tau_n$ of individuals in the portfolio are independent of the financial market and the number of deaths in the portfolio, and conditioned on the filtration generated by Λ, they are independent and identically distributed with survival distribution

$$\mathbb{P}(\tau_i > t|\sigma(\lambda(u), u \leq T)) = e^{-\int_0^t \lambda(s)ds}, \quad i = 1, ..., n,$$

(B3) the functions $h, g, f : [0, T] \times (0, \infty) \to \mathbb{R}$ are product measurable.

Under (2.8) each policyholder in the portfolio is entitled to three types of payments h, g, f. Firstly, there are claims payable continuously, at a rate h, as long as an insured person is alive. These could be benefits, in the case of positive cashflows, or premiums, in the case of negative cashflows. Secondly, there is a benefit in the amount of g payable at the time of death of an insured person (at the time of jump of Y). Thirdly, there is a terminal claim in the amount of f payable provided that an insured person is still alive at the terminal time T. The liabilities' pay-offs are linked to the value of the risky asset. The examples include payments from unit-linked products, variable annuities, structured products, etc., which constitute a major part of the insurance market.

We remark that the assumptions **(B1)**-**(B2)** coincide with the definition of the counting process Y with (ξ, Q) as in (2.9), see [9], [10]. The assumptions **(B1)**-**(B2)** clearly hold true when dealing with population mortality, and yield $[L, Y] = 0$ (by independence and stochastic continuity of L and Y) and **(A9)** is satisfied. Notice that in reality the benefits paid at death times are of the form $\sum g(s, S(s))\triangle Y(s)$. However, $\sum g(s, S(s))\triangle Y(s) = \sum g(s, S(s-))\triangle Y(s)$, $\mathbb{P}$-a.s., since Y and S never jump together, and the second sum can be represented as a well-defined stochastic integral in Itô sense. By investigating (2.8)-(2.9) we take into account not only the unsystematic mortality risk generated by Y but also the systematic mortality risk generated by Λ, and we can consider for example annuities under longevity risk as well. Technically, there is no problem in linking the benefits' values to the value of the process Λ and derive explicit hedging strategies for mortality-dependent products like guaranteed annuity options. However, we do not take up this (feasible but tedious) extension and recall that quadratic hedging of a guaranteed annuity option under longevity risk is considered for example in Delong et al. (2008), to which we refer the interested reader.

2.2.2 The second special case: the stream of liabilities in non-life insurance

In non-life insurance mathematics cumulative loss processes are usually based on a compound Poisson process and its generalizations, see Bening, Korolev (2002). We consider the payment process P given by

$$P(t) \quad = \quad \int_0^t \int_{\mathbb{R}} g_0(s, S(s-), y) N(ds, dy), \quad 0 \le t \le T, \tag{2.10}$$

with the step process Y under the characteristics

$$\begin{aligned} \xi(t) &= \lambda(t), \quad 0 \le t \le T, \\ Q(t, dy) &= q(dy), \quad 0 \le t \le T, \end{aligned} \tag{2.11}$$

where we assume that

(C1) the claim intensity is independent of the financial market and the number of claims in the portfolio, i.e. Λ is independent of L, Y,

(C2) the step process Y is a compound Cox process, independent of L, with jump sizes distributed according to a (non-random) probability measure q on $\mathbb{R}$ with $q(\{0\}) = 0$,

(C3) the function $g_0 : [0, T] \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is product measurable.

The independence assumptions **(C1)**-**(C2)** seem to be reasonable in many situations. Recall that **(C2)** implies that the condition **(A9)** is satisfied. Under our assumptions, the insurance claims (2.10) occur at a rate independent of the financial market (at the time when Y jumps), but the severity of a claim depends, via g_0, both on the value of an auxiliary independent random variable distributed according to q, as in the classical risk theory, and on the value of the tradeable asset S. The fundamental example of P satisfying (2.10)-(2.11) is an independent compound Poisson process, which is the cornerstone of the non-life insurance risk theory. However, more interesting examples fit into our framework. In non-life insurance, systematic claim intensity risk also constitutes an important risk factor. In car insurance or in fire insurance, seasonal and random variations of a portfolio claim intensity are clearly observable and cause unpredictable fluctuations in the number of claims. In order to take into account this systematic risk of insurance claims, we could apply an independent compound Cox process with stochastic intensity. A more exotic example of the claim (2.10), depending also on a financial asset, includes a stop-loss reinsurance contract covering insurance risks with a barrier/strike determined by an independent financial index, see Møller (2003). Such contracts provide a coverage for losses due to large fluctuations within insurance portfolios and for adverse developments in financial markets. This kind of instruments are gaining more interest in insurance (reinsurance) markets and are effective risk transfer tools.

We realize that the independence assumptions in non-life insurance **(C1)-(C2)** might be more often questioned, compared to the independence assumptions in life insurance **(B1)-(B2)**, which hold almost always. We point out that we can assume more generally that ξ depends on the risky asset S or on the compound Cox process Y, and consider the linear models (and slight modifications as well)

$$\begin{aligned} \xi(t) &= \rho S(t-) + \lambda(t), \quad 0 \le t \le T, \\ \xi(t) &= \rho Y(t-) + \lambda(t), \quad 0 \le t \le T, \end{aligned} \tag{2.12}$$

with a correlation factor ρ. The dependence structures (2.12) extend the area of applications to cases when a portfolio claim intensity is correlated with a tradeable asset, see Ankirchner, Imkeller (2008), or to the cases of contagion models with Polya-type phenomena (common in risk theory). It is not a problem to derive explicit results under the relations (2.12), which weaken the assumptions **(C1)-(C2)**. A similar reasoning and calculations as in Section 6 can be followed.

We believe that (2.8)-(2.9) and (2.10)-(2.11) are the most interesting insurance payment processes to investigate, as they take into account the most important risks to which an insurance company is exposed. In what follows, we call (2.8)-(2.9) and (2.10)-(2.11) equity-linked payment processes with independent (unsystematic and systematic) insurance risk.

3 The optimization problem

Let us consider a wealth process of an insurer, $X^\pi := (X^\pi(t), 0 \le t \le T)$, which arises as the result of dynamic trading, according to a strategy π, in the financial market and covering the claims H and G. The dynamics of X^π is given by the stochastic differential equation

$$\begin{aligned} dX^\pi(t) &= \pi(t)\Big(\mu dt + \sigma dW(t) + \int_{\mathbb{R}} z\tilde{M}(dt,dz)\Big) + \big(X^\pi(t-) - \pi(t)\big)rdt \\ & \quad -H(t)dt - \int_{\mathbb{R}} G(t,y)N(dt,dy), \\ X(0) &= x_0 > 0, \end{aligned} \tag{3.1}$$

where π denotes an amount of wealth invested in the risky asset S and x_0 denotes an initial capital including premiums collected at the inception of a contract.

We define admissible investment strategies.

Definition 3.1. *A strategy* $\pi := (\pi(t), 0 \le t \le T)$ *is called admissible, written* $\pi \in \mathcal{A}$, *if it satisfies the conditions:*

1. $\pi : [0,T] \times \Omega \to \mathbb{R}$ *is a predictable process,*
2. $\mathbb{E}\big[\int_0^T |\pi(t)|^2 dt\big] < \infty$,
3. *there exists a unique càdlàg solution* X^π *of* (3.1) *on* $[0,T]$.

In this paper we investigate the following asset-liability optimization problem:

$$\inf_{\pi \in \mathcal{A}} \mathbb{E}\big[\theta \int_0^T \big(X^\pi(t) - \psi(t)\big)^2 dt + \big(X^\pi(T) - \psi(T)\big)^2\big], \tag{3.2}$$

where

(A10) $\psi : \Omega \times [0,T] \to [0,\infty)$ is an $\mathbb{F}$-adapted, product measurable and a.s. finite process, independent of π, satisfying the integrability condition $\sup_{t\in[0,T]} \mathbb{E}[|\psi(t)|^2] < \infty$.

The optimization problem (3.2) is a quadratic control problem with a terminal penalty and a running cost. Such criterions have already been considered in insurance, see Detemple, Rindisbacher (2008), Gerrard et al. (2004), and finance, see Kohlmann, Peisl (2000). We do not mention here the papers in which quadratic control problems with a terminal objective only are considered, referring the reader to the Introduction. The parameter θ attaches a weight to the running cost compared to the terminal penalty. Under the functional (3.2), the investment strategy is chosen in such a way that the resulting wealth process is as close as possible, in the mean-square error sense, to the target process ψ. The process ψ could represent required solvency constraints to be fulfilled in the future and the insurer's expectations concerning the profit. Notice that ψ includes the terminal pay-off F. An example of the target, which will be investigated in detail in Section 6, can be

(D1) $\psi(t) = \phi(t) + R(t), \quad 0 \le t \le T$,

where the continuous function $\phi : [0,T] \to [0,\infty)$ describes the required profit which should accumulate until time t and $R := (R(t), 0 \le t \le T)$ denotes a reserve which must be kept at time t, under statutory requirements, in order to meet future contractual obligations. The consideration of the running cost of this form is very reasonable as it means that the optimally controlled assets should generate values satisfying the desired profit as well as the required solvency constraints and stabilizes deviations of the wealth process during the whole duration of an insurance contract.

By rewriting (3.1) in the form

$$\begin{aligned} X^{\pi}(t) &= x_0e^{rt} + \int_0^t (\mu - r)e^{r(t-s)}\pi(s)ds \\ &\quad + \int_0^t \sigma e^{r(t-s)}\pi(s)dW(s) + \int_0^t\int_{\mathbb{R}} e^{r(t-s)}\pi(s)z\tilde{M}(ds,dz) \\ &\quad - \int_0^t e^{r(t-s)}H(s)ds - \int_0^t\int_{\mathbb{R}} e^{r(t-s)}G(s,y)N(ds,dy), \quad 0 \le t \le T, \end{aligned}$$

we can deduce, similarly to (2.7), that for an admissible strategy $\pi \in \mathcal{A}$ and under **(A2)**, **(A6)**, **(A7)** the wealth process is square integrable:

$$\mathbb{E}\big[\sup_{t\in[0,T]} |X^{\pi}(t)|^2\big] < \infty,$$

which together with **(A10)** shows that the optimization criterion (3.2) is well-defined.

To end this section let us define some topological spaces of processes. They become crucial when proving existence of a solution to (3.2).

Definition 3.2. *1. Let $\mathbb{S}^2(\mathbb{R})$ denote the space of càdlàg processes $U : \Omega \times [0,T] \to \mathbb{R}$ satisfying*

$$\mathbb{E}\big[\sup_{t\in[0,T]} \big|U(t)\big|^2\big] < \infty.$$

2. Let $\mathbb{H}^2(\mathbb{R})$ denote the space of predictable processes $U : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\int_0^T \big|U(t)\big|^2 dt\big] < \infty.$$

3. Let $\mathbb{H}^2_M(\mathbb{R})$ denote the space of predictable processes $U : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\int_0^T \int_{\mathbb{R}} \big|U(t,z)\big|^2 \nu(dz)dt\big] < \infty.$$

4. Let $\mathbb{H}^2_N(\mathbb{R})$ denote the space of predictable processes $U : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\int_0^T \int_{\mathbb{R}} \big|U(t,y)\big|^2 \xi(t)Q(t,dy)dt\big] < \infty.$$

The spaces $\mathbb{S}^2(\mathbb{R})$,$\mathbb{H}^2(\mathbb{R})$,$\mathbb{H}^2_M(\mathbb{R})$ *and* $\mathbb{H}^2_N(\mathbb{R})$ *are endowed with the norms*

$$\begin{aligned}
\|U\|^2_{\mathbb{S}^2} &= \mathbb{E}\big[\sup_{t\in[0,T]} \big|U(t)\big|^2\big], \\
\|U\|^2_{\mathbb{H}^2} &= \mathbb{E}\big[\int_0^T e^{\varrho t}\big|U(t)\big|^2 dt\big], \\
\|U\|^2_{\mathbb{H}^2_M} &= \mathbb{E}\big[\int_0^T \int_{\mathbb{R}} e^{\varrho t}\big|U(t,z)\big|^2 \nu(dz)dt\big], \\
\|U\|^2_{\mathbb{H}^2_N} &= \mathbb{E}\big[\int_0^T \int_{\mathbb{R}} e^{\varrho t}\big|U(t,y)\big|^2 \xi(t)Q(t,dy)dt\big],
\end{aligned}$$

with some $\varrho > 0$*.*

4 The general solution of the optimization problem

In this section we solve the stated asset-liability optimization problem (3.2) by applying the theory of backward stochastic differential equations (BSDEs). For an introduction to BSDEs we refer the reader to El Karoui et al. (1997).

In order to apply the theory of BSDEs we need a martingale representation theorem. Let us assume that under $(\mathbb{P},\mathbb{F})$ the weak property of predictable representation holds, see Chapter XII.2 in He et al. (1992), i.e.

(A11) every $(\mathbb{P},\mathbb{F})$ local martingale $\mathcal{Z}$ null at zero has a representation

$$\begin{aligned}
\mathcal{Z}(t) &= \int_0^t \beta(s)dW(s) + \int_0^t \gamma(s)dB(s) \\
&\quad + \int_0^t \int_{\mathbb{R}} \kappa(s,z)\tilde{M}(ds,dz) + \int_0^t \int_{\mathbb{R}} \eta(s,y)\tilde{N}(ds,dy), \quad 0 \le t \le T,
\end{aligned}$$

with predictable processes $(\beta,\gamma,\kappa,\eta)$ integrable in the sense of Itô calculus.

In particular, if the local martingale $\mathcal{Z}$ is square integrable, then $(\beta,\gamma,\kappa,\eta) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})_M \times \mathbb{H}^2_N(\mathbb{R})$ and the representation is unique in $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})_M \times \mathbb{H}^2_N(\mathbb{R})$. Cases when the representation **(A11)** holds are exhibited in Becherer (2006), and the key example includes martingales with respect to the product natural filtration generated by independent Lévy processes and step processes. We point out that **(A11)** is only needed if we assume that the claims (2.6) are driven by a general random measure, whereas it holds if we assume that the claims are driven by a step process. By a suitable change of measure, one can construct dependent processes L and Y defined on the space $(\mathbb{P},\mathbb{F})$ on which the representation **(A11)** holds. The details of the construction are omitted and we refer to Examples 2.1 and 4.5 in Becherer (2006).

Derivation of the optimal solution is based on a good guess. How to derive the guess, a candidate for the solution, becomes clearer when studying the forthcoming proof of optimality. First, consider the ordinary differential equation on $0 \leq t \leq T$

$$\begin{aligned} \frac{d\tilde{p}(t)}{dt} &= -\tilde{p}(t)\Big(2r - \frac{(\mu - r)^2}{\sigma^2 + \int_{\mathbb{R}} z^2 \nu(dz)}\Big) - \theta, \\ \tilde{p}(T) &= 1, \end{aligned} \tag{4.1}$$

and the backward stochastic differential equation on $0 \leq t \leq T$

$$\begin{aligned} dp(t) = &-\bigg(p(t-)r - 2\theta\psi(t) - 2\tilde{p}(t)\big(H(t) + \int_{\mathbb{R}} G(t,y)\xi(t)Q(t,dy)\big) \\ &- \frac{\mu - r}{\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)}\big(p(t-)(\mu - r) + \beta(t)\sigma + \int_{\mathbb{R}} \kappa(t,z)z\nu(dz)\big)\bigg)\, dt \\ &+\beta(t)dW(t) + \gamma(t)dB(t) + \int_{\mathbb{R}} \kappa(t,z)\tilde{M}(dt,dz) + \int_{\mathbb{R}} \eta(t,y)\tilde{N}(dt,dy), \\ p(T) = &-2\psi(T). \end{aligned} \tag{4.2}$$

Clearly, (4.1) has a unique solution $\tilde{p}$ that is strictly positive, continuously differentiable on $\mathbb{R}$ and uniformly bounded above and bounded away from zero. Moreover, see the Appendix, under **(A1)-(A11)**, there exists a unique solution $(p, \beta, \gamma, \kappa, \eta) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_M(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ of (4.2) and the solution satisfies, for $t \in [0,T]$, the fixed point equation

$$\begin{aligned} &p(t) \\ &= \mathbb{E}\bigg[-2\psi(T) + \int_t^T \Big(p(s)r - 2\theta\psi(s) - 2\tilde{p}(s)\big(H(s) + \int_{\mathbb{R}} G(s,y)\xi(s)Q(s,dy)\big) \\ &\quad - \frac{\mu - r}{\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)}\big(p(s)(\mu - r) + \beta(s)\sigma + \int_{\mathbb{R}} \kappa(s,z)z\nu(dz)\big)\Big)\, ds\Big|\mathcal{F}_t\bigg]. \end{aligned} \tag{4.3}$$

We now state the main result of this paper.

Theorem 4.1. *Assume that **(A1)-(A11)** hold. The investment strategy*

$$\begin{aligned} \hat{\pi}(t) &= -\frac{\mu - r}{\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)} X^{\hat{\pi}}(t-) \\ &\quad - \frac{p(t-)(\mu - r) + \beta(t)\sigma + \int_{\mathbb{R}} \kappa(t,z)z\nu(dz)}{2\tilde{p}(t)(\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz))}, \quad 0 \leq t \leq T, \end{aligned} \tag{4.4}$$

with $X^{\hat{\pi}}$ being the wealth process (3.1) *under the strategy $\hat{\pi}$, and $\tilde{p}$ and (p, β, κ) being the solutions of the ordinary differential equation* (4.1) *and the backward stochastic differential equation* (4.2)*, is the optimal investment strategy for the optimization problem* (3.2).

Proof:
We prove that our guess, consisting of complicated equations (4.1), (4.2) and the strategy (4.4), leads indeed to the optimal solution. The idea of deriving the candidate solution and verification is analogous to Lim (2005) or Øksendal, Hu (2008) and is based on the method of completing squares.
Consider the following equations on $0 \leq t \leq T$

$$\begin{aligned} d\tilde{p}(t) &= -\tilde{\alpha}(t)dt, \\ \tilde{p}(T) &= 1, \\ dp(t) &= -\alpha(t)dt + \beta(t)dW(t) + \gamma(t)dB(t) + \int_{\mathbb{R}} \kappa(t,z)\tilde{M}(dt,dz) + \int_{\mathbb{R}} \eta(t,y)\tilde{N}(dt,dy), \\ p(T) &= -2\psi(T), \end{aligned}$$

with generators $\tilde{\alpha}, \alpha$ defined according to (4.1) and (4.2). In fact we start with arbitrary functions $\alpha, \tilde{\alpha}$ and arrive at the specific form of the generators at the very end when completing squares, see Øksendal, Hu (2008) for the full explanation. Recall that the solution $\tilde{p}$ is given by (4.1) and the solution $(p, \beta, \gamma, \kappa, \eta)$ is given by Lemma 8.1. Let π denote an arbitrary admissible strategy, $\pi \in \mathcal{A}$. By applying the Itô formula we can derive

$$\begin{aligned} d\big(\tilde{p}(t)(X^\pi(t))^2\big) = \tilde{p}(t)\Big(&2X^\pi(t-)\pi(t)\big(\mu dt + \sigma dW(t) + \int_{\mathbb{R}} z\tilde{M}(dt,dz)\big) \\ &+2X^\pi(t-)\big(X^\pi(t-) - \pi(t)\big)rdt - 2X^\pi(t-)H(t)dt - \int_{\mathbb{R}} 2X^\pi(t-)G(t,y)N(dt,dy) \\ &+\pi^2(t)\sigma^2 dt + \int_{\mathbb{R}} \pi^2(t)z^2 M(dt,dz) + \int_{\mathbb{R}} G^2(t,y)N(dt,dy)\Big) \\ &-(X^\pi(t-))^2\tilde{\alpha}(t)dt \end{aligned}$$

and

$$\begin{aligned} d\big(p(t)X^\pi(t)\big) = p(t-)\Big(&\pi(t)\big(\mu dt + \sigma dW(t) + \int_{\mathbb{R}} z\tilde{M}(dt,dz)\big) \\ &+\big(X^\pi(t-) - \pi(t)\big)rdt - H(t)dt - \int_{\mathbb{R}} G(t,y)N(dt,dy)\Big) \\ &+X^\pi(t-)\Big(-\alpha(t)dt + \beta(t)dW(t) + \gamma(t)dB(t) + \int_{\mathbb{R}} \kappa(t,z)\tilde{M}(dt,dz) \\ &+\int_{\mathbb{R}} \eta(t,y)\tilde{N}(dt,dy)\Big) + \beta(t)\pi(t)\sigma dt + \int_{\mathbb{R}} \pi(t)z\kappa(t,z)M(dt,dz) \\ &-\int_{\mathbb{R}} G(t,y)\eta(t,y)N(dt,dy). \end{aligned}$$

Notice that the stochastic integrals with respect to $W, B, \tilde{M}, \tilde{N}$ are locally square integrable local martingales (as X^π, p are càdlàg and $\tilde{p}$ is bounded, $\pi, \beta, \gamma, \kappa, \eta$ are square integrable), whereas the stochastic integrals with respect to N, M are adapted processes with locally integrable variations (as X^π, p are càdlàg and $\tilde{p}$ is bounded, π, G, κ, η are square integrable), hence their compensated integrals are local martingales, see Theorem 11.21 in He et al. (1992). Let $(\tau_n)_{n\geq 1}$ denote a localizing sequence for the stochastic integrals and $\tau_n \to T$, $\mathbb{P}$-a.s.. We obtain

$$\begin{aligned}\mathbb{E}\big[\tilde{p}(\tau_n)(X^\pi(\tau_n))^2\big] = \tilde{p}(0)x_0^2 + \mathbb{E}\Big[\int_0^{\tau_n} \Big\{\tilde{p}(s)\Big(2X^\pi(s-)\pi(s)(\mu - r)\\ +2(X^\pi(s-))^2 r - 2X^\pi(s-)H(s) - \int_{\mathbb{R}} 2X^\pi(s-)G(s,y)\xi(s)Q(s,dy)\\ +\pi^2(s)\sigma^2 + \int_{\mathbb{R}} \pi^2(s)z^2\nu(dz)ds + \int_{\mathbb{R}} G^2(s,y)\xi(s)Q(s,dy)\Big)\\ -(X^\pi(s-))^2\tilde{\alpha}(s)\Big\}ds\Big],\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\mathbb{E}\big[p(\tau_n)X^\pi(\tau_n)\big] = p(0)x_0 + \mathbb{E}\Big[\int_0^{\tau_n} \Big\{p(s-)\Big(\pi(s)(\mu - r) + X^\pi(s-)r\\ -H(s) - \int_{\mathbb{R}} G(s,y)\xi(s)Q(s,dy)\Big) - X^\pi(s-)\alpha(s) + \beta(s)\pi(s)\sigma\\ + \int_{\mathbb{R}} \pi(s)z\kappa(s,z)\nu(dz) - \int_{\mathbb{R}} G(s,y)\eta(s,y)\xi(s)Q(s,dy)\Big\}ds\Big].\end{aligned} \tag{4.6}$$

Recall that $\tilde{p}$ is uniformly bounded, $\mathbb{E}[\sup_{t\in[0,T]} |p(t)|^2] < \infty$ and $\mathbb{E}[\sup_{t\in[0,T]} |X^\pi(t)|^2] < \infty$ for any $\pi \in \mathcal{A}$, so $\tilde{p}(\tau_n)(X^\pi(\tau_n))^2$ and $p(\tau_n)X^\pi(\tau_n)$ are bounded uniformly in n by square integrable random variables. We let $n \to \infty$ and by applying Lebesgue's dominated convergence theorem on both sides of (4.5) and (4.6) we obtain expressions for $\mathbb{E}[\tilde{p}(T)(X^\pi(T))^2]$ and $\mathbb{E}[p(T)X^\pi(T)]$. The straightforward justification for applying Lebesgue's theorem under the assumptions **(A1)-(A10)** is left to the reader.
As $\tilde{p}(T) = 1$ and $p(T) = -2\psi(T)$ we can write

$$\begin{aligned}&\mathbb{E}\Big[\theta\int_0^T \big(X^\pi(t) - \psi(t)\big)dt + (X^\pi(T) - \psi(T))^2\Big]\\ &\quad= \mathbb{E}\Big[\theta\int_0^T \big(X^\pi(t) - \psi(t)\big)dt\Big] + \mathbb{E}\Big[\tilde{p}(T)(X^\pi(T))^2\Big] + \mathbb{E}\Big[p(T)X^\pi(T)\Big] + \mathbb{E}\Big[\psi^2(T))\Big]\end{aligned}$$

Substituting the expressions (4.5), (4.6) with $\tau_n \to T$, grouping terms to obtain the complete square of

$$\Bigg(\pi(t) + \frac{\mu - r}{\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)}X^\pi(t-) + \frac{p(t-)(\mu - r) + \beta(t)\sigma + \int_{\mathbb{R}}\kappa(t,z)z\nu(dz)}{2\tilde{p}(t)(\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz))}\Bigg)^2$$

and using the definitions of the generators $\tilde{\alpha}, \alpha$, appearing in (4.1) and (4.2) (at this point we obtain their precise forms), we derive after some tedious calculations that

$$\begin{aligned}
&\mathbb{E}\Big[\theta \int_0^T \big(X^\pi(t) - \psi(t)\big)dt + (X^\pi(T) - \psi(T))^2\Big] \\
&= \tilde{p}(0)x_0^2 + p(0)x_0 + \mathbb{E}[\zeta] + \mathbb{E}\Big[\int_0^T \tilde{p}(t)\big(\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)\big) \\
&\times \left(\pi(t) + \frac{\mu - r}{\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)} X^\pi(t-) + \frac{p(t-)(\mu - r) + \beta(t)\sigma + \int_{\mathbb{R}} \kappa(t,z)z\nu(dz)}{2\tilde{p}(t)(\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz))}\right)^2 dt\Big]
\end{aligned}$$

where ζ is a random variable independent of π. Since $\hat{\pi}$ is admissible, see the Appendix, the optimality of $\hat{\pi}$ is now proved. □

5 The case of martingale dynamics or upward bounded jumps of the risky asset

In this section we show that in some special cases it is possible to construct the unique solution of (4.2) in a more explicit form. We remark that, in the general setting of the previous section, solving the fixed point equation (4.3) or the backward stochastic differential equation (4.2) requires numerical methods. We refer the reader to Ma et al. (2002) for an introduction to numerical methods for BSDEs.

We still work under the conditions **(A1)**-**(A11)** and let us assume additionally that

(A12) the measure ν is defined on a subset $A \subset (-1, \infty)$ such that $\frac{(\mu - r)z}{\sigma^2 + \int_A z^2\nu(dz)} < 1$ for all $z \in A$.

There are two possible forms of the set A. If $\mu > r$ then it is easy to see that A is an interval $(-1, \bar{z}]$ with a finite $\bar{z} > 0$. In this case the assumption **(A12)** requires that the jumps of the Lévy process L must be bounded from above. This holds for spectrally negative Lévy processes. The assumption that a risky-asset's price follows a spectrally negative Lévy process might be reasonable. Such a conservative assumption would mean that we consider unbounded financial crashes (downward jumps) but bounded financial gains (upward jumps) and it might be used by a risk-averse risk manager. We point out that a gain/loss asymmetry is an empirical fact observed for stock prices' movements, see Chapter 7.1 in Cont, Tankov (2004). If $\mu = r$ then the set A coincides trivially with $(-1, \infty)$. Recall that the relation $\mu = r$ is satisfied when the real-world measure $\mathbb{P}$ is already a martingale measure, i.e. a measure under which the discounted risky-asset's price process is a martingale. If this were the case then the optimization functional (3.2) would correspond to minimizing the square error under a martingale measure which is also often applied when dealing with hedging, see Ankirchner, Imkeller (2008),Dahl et al. (2007), Dahl, Møller (2006), Møller (2001), Vandaele, Vanmaele (2008). Optimization under a martingale measure is interesting and important in its own right and sometimes in practice it may be the only feasible way to apply quadratic hedging, see Chapter 10.4.3 in Cont, Tankov (2004). We remark that there exist infinitely many martingale measures in our model. Finally, we point out that we still integrate over $\mathbb{R}$ and recall that the integration is restricted to the subset A.

Consider the independent processes $Z^W := (Z^W(t), 0 \leq t \leq T), Z^M := (Z^M(t), 0 \leq t \leq T)$ defined by the stochastic exponentials

$$\frac{dZ^W(t)}{Z^W(t-)} = -\frac{(\mu - r)\sigma}{\sigma^2 + \int_A z^2 \nu(dz)} dW(t), \quad Z^W(0) = 1, \tag{5.1}$$

$$\frac{dZ^M(t)}{Z^M(t-)} = -\frac{(\mu - r)}{\sigma^2 + \int_A z^2 \nu(dz)} z\tilde{M}(dt, dz), \quad Z^M(0) = 1, \tag{5.2}$$

and the measure $\mathbb{Q}$ defined by Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = Z^W(T) Z^M(T). \tag{5.3}$$

It is obvious that Z^W is a $\mathbb{P}$-martingale possessing all moments. By applying a localizing sequence, Cauchy-Schwarz' inequality, Gronwall's inequality and Fatou's lemma we can derive, see (8.1) in the Appendix, that

$$\sup_{t \in [0,T]} \mathbb{E}[|Z^M(t)|^2] < \infty,$$

which yields

$$\mathbb{E}\big\{[Z^M, Z^M]_T\big\} = \mathbb{E}\Big\{\int_0^T \int_A |Z^M(t)|^2 \Big(\frac{\mu - r}{\sigma^2 + \int_A z^2 \nu(dz)}\Big)^2 z^2 \nu(dz) dt\Big\} < \infty,$$

and we conclude that a $\mathbb{P}$-local martingale Z^M is a square integrable $\mathbb{P}$-martingale, hence $\mathbb{Q}$ is an equivalent probability measure, $\mathbb{Q} \sim \mathbb{P}$, with the corresponding square integrable density process $(Z(t), 0 \leq t \leq T)$ satisfying

$$\frac{dZ(t)}{Z(t-)} = -\frac{(\mu - r)\sigma}{\sigma^2 + \int_A z^2 \nu(dz)} dW(t) - \frac{\mu - r}{\sigma^2 + \int_A z^2 \nu(dz)} z\tilde{M}(dt, dz), \quad Z(0) = 1.$$

Define for $0 \leq t \leq T$

$$\begin{aligned} dW^{\mathbb{Q}}(t) &= dW(t) + \frac{(\mu - r)\sigma}{\sigma^2 + \int_A z^2 \nu(dz)} dt, \\ \tilde{M}^{\mathbb{Q}}(dt, dz) &= M(dt, dz) - \big(1 - \frac{\mu - r}{\sigma^2 + \int_A z^2 \nu(dz)} z\big)\nu(dz) dt, \end{aligned}$$

and rewrite the equation (4.2), on $0 \le t \le T$, under the measure $\mathbb{Q}$:

$$
\begin{aligned}
dp(t) &= -\Big(p(t-)r - 2\theta\psi(t) - 2\tilde{p}(t)\big(H(t) + \int_{\mathbb{R}} G(t,y)\xi(t)Q(t,dy)\big) \\
&\quad - \frac{\mu - r}{\sigma^2 + \int_A z^2\nu(dz)} p(t-)(\mu - r)\Big)\, dt \\
&\quad + \beta(t)dW^{\mathbb{Q}}(t) + \gamma(t)dB(t) + \int_{\mathbb{R}} \kappa(t,z)\tilde{M}^{\mathbb{Q}}(dt,dz) + \int_{\mathbb{R}} \eta(t,y)\tilde{N}(dt,dy) \\
p(T) &= -2\psi(T). \qquad (5.4)
\end{aligned}
$$

Recall that we consider a solution $(p,\beta,\gamma,\kappa,\eta) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_M(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ of (4.2). We notice that the stochastic integrals in (4.2) driven by $W, B, \tilde{M}, \tilde{N}$ are martingales under $\mathbb{P}$ and the stochastic integrals in (5.4) driven by $W^{\mathbb{Q}}, B, \tilde{M}^{\mathbb{Q}}, \tilde{N}$ are local martingales under $\mathbb{Q}$. This claim follows form the Girsanov-Meyer Theorem, see Theorem III.40 in Protter (2004). In particular, $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion, $\tilde{M}^{\mathbb{Q}}$ is a $\mathbb{Q}$-compensated random measure (corresponding to some Lévy process), B remains a Brownian motion under $\mathbb{Q}$ and the compensator for N under $\mathbb{Q}$ remains the same ϑ.

Let us denote

$$
\delta = r - \frac{(\mu - r)^2}{\sigma^2 + \int_A z^2\nu(dz)}.
$$

By the change of variables $\bar{p}(t) = p(t)^{\delta t}, \bar{\beta}(t) = \beta(t)e^{\delta t}, \bar{\gamma}(t) = \gamma(t)e^{\delta t}, \bar{\kappa}(t,z) = \kappa(t,z)e^{\delta t}, \bar{\eta}(t,z) = \eta(t,z)e^{\delta t}$ we obtain the equivalent form of (4.2) on $0 \le t \le T$

$$
\begin{aligned}
d\bar{p}(t) &= -\Big(-2e^{\delta t}\theta\psi(t) - 2e^{\delta t}\tilde{p}(t)\big(H(t) + \int_{\mathbb{R}} G(t,y)\xi(t)Q(t,dy)\big)\Big)\, dt \\
&\quad + \bar{\beta}(t)dW^{\mathbb{Q}}(t) + \bar{\gamma}(t)dB(t) + \int_{\mathbb{R}} \bar{\kappa}(t,z)\tilde{M}^{\mathbb{Q}}(dt,dz) + \int_{\mathbb{R}} \bar{\eta}(t,y)\tilde{N}(dt,dy), \\
\bar{p}(T) &= -2\psi(T)e^{\delta T}. \qquad (5.5)
\end{aligned}
$$

As the weak property of predictable representation **(A11)** is assumed to hold for all $\mathbb{P}$-local martingales, it also holds for all $\mathbb{Q}$-local martingales, see Theorem 13.22 in He et al. (1992). We can state the main conclusion of this section.

Theorem 5.1. *Assume that **(A1)-(A12)** hold and the uniformly integrable $\mathbb{Q}$-martingale has the following representation:*

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[-2\psi(T)e^{\delta T} - 2\int_0^T e^{\delta s}\Big(\theta\psi(s) + \tilde{p}(s)\big(H(s) + \int_{\mathbb{R}} G(s,y)\xi(s)Q(s,dy)\big)\Big)ds\Big|\mathcal{F}_t\Big] \\
&\quad -\mathbb{E}^{\mathbb{Q}}\Big[-2\psi(T)e^{\delta T} - 2\int_0^T e^{\delta s}\Big(\theta\psi(s) + \tilde{p}(s)\big(H(s) + \int_{\mathbb{R}} G(s,y)\xi(s)Q(s,dy)\big)\Big)ds\Big] \\
&= \int_0^t \bar{\beta}(t)dW^{\mathbb{Q}}(s) + \int_0^t \bar{\gamma}(t)dB(s) \\
&\quad + \int_0^t\int_{\mathbb{R}} \bar{\kappa}(s,z)\tilde{M}^{\mathbb{Q}}(ds,dz) + \int_0^t\int_{\mathbb{R}} \bar{\eta}(s,y)\tilde{N}(ds,dy), \quad 0 \le t \le T, \mathbb{Q} - a.s.,
\end{aligned}
$$

where the equivalent measure $\mathbb{Q}$ *is defined in* (5.3). *If* $(\bar{\beta}, \bar{\gamma}, \bar{\kappa}, \bar{\eta}) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_M(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ *under the measure* $\mathbb{P}$ *then* $(p, \beta, \gamma, \kappa, \eta)$ *defined for* $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ *by*

$$\begin{aligned} p(t) &= \mathbb{E}^{\mathbb{Q}}\Big[-2\psi(T)e^{\delta(T-t)} \\ &\quad -2\int_t^T e^{\delta(s-t)}\Big(\theta\psi(s) + \tilde{p}(s)\big(H(s) + \int_{\mathbb{R}} G(s,y)\xi(s)Q(s,dy)\big)\Big)ds\Big|\mathcal{F}_t\Big] \\ \beta(t) &= \bar{\beta}(t)e^{-\delta t}, \\ \gamma(t) &= \bar{\gamma}(t)e^{-\delta t}, \\ \kappa(t,z) &= \bar{\kappa}(t,z)e^{-\delta t}, \\ \eta(t,y) &= \bar{\eta}(t,y)e^{-\delta t}, \end{aligned}$$

is the unique solution of the backward stochastic differential equation (4.2) *in* $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_M(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$.

We point out that by Cauchy-Schwarz' inequality, square integrability of Z, boundedness of $\tilde{p}$ and the assumptions **(A6)**, **(A7)**, **(A10)**, we obtain the integrability of

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\Big[\big| -2\psi(T)e^{\delta T} - 2\int_0^T e^{\delta t}\Big(\theta\psi(t) + \tilde{p}(t)\big(H(t) + \int_{\mathbb{R}} G(t,y)\xi(t)Q(t,dy)\big)\Big)dt\big|\Big] \\ &\leq K\Bigg(\mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)^2\Big] \\ &\quad \times \mathbb{E}^{\mathbb{P}}\Big[|\psi(T)|^2 + \int_0^T \big(|\psi(t)|^2 + |\tilde{p}(t)|^2\big(|H(t)|^2 + \int_{\mathbb{R}} |G(t,y)\xi(t)|^2 Q(t,dy)\big)\big)dt\Big]\Bigg)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

which implies that the martingale

$$\mathbb{E}^{\mathbb{Q}}\Big[-2\psi(T)e^{\delta T} - 2\int_0^T e^{\delta s}\Big(\theta\psi(s) + \tilde{p}(s)\big(H(s) + \int_{\mathbb{R}} G(s,y)\xi(s)Q(s,dy)\big)\Big)ds\Big|\mathcal{F}_t\Big]$$

is indeed a uniformly integrable $\mathbb{Q}$-martingale. Moreover, notice that again by Cauchy-Schwarz' inequality, the submartingale property of Z^2, boundedness of $\tilde{p}$, the assumptions **(A6)**, **(A7)**, **(A10)** and the property of conditional expectations we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[|p(t)|^2\right] \\
&\quad\leq\; \mathbb{E}^{\mathbb{P}}\Big\{\mathbb{E}^{\mathbb{Q}}\Big[\Big| - 2\psi(T)e^{\delta T} \\
&\qquad -2\int_t^T e^{\delta t}\Big(\theta\psi(t)+\tilde{p}(t)\big(H(t)+\int_{\mathbb{R}} G(t,y)\xi(t)Q(t,dy)\big)\Big)dt\Big|\big|\mathcal{F}_t\Big]\Big\}^2 \\
&\quad\leq\; K\mathbb{E}^{\mathbb{P}}\Big\{\mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{Z(T)}{Z(t)}\Big)^2|\mathcal{F}_t\Big] \\
&\qquad \times\mathbb{E}^{\mathbb{P}}\Big[|\psi(T)|^2+\int_0^T\Big(|\psi(t)|^2+|\tilde{p}(t)|^2\big(|H(t)|^2+\int_{\mathbb{R}}|G(t,y)\xi(t)|^2Q(t,dy)\big)\Big)dt\big|\mathcal{F}_t\Big]\Big\} \\
&\quad\leq\; K\mathbb{E}^{\mathbb{P}}\Big[|\psi(T)|^2+\int_0^T\Big(|\psi(t)|^2+|\tilde{p}(t)|^2\big(|H(t)|^2+\int_{\mathbb{R}}|G(t,y)\xi(t)|^2Q(t,dy)\big)\Big)dt\Big] \\
&\quad<\; \infty,
\end{aligned}
$$

and by applying Doob's martingale inequality to (4.3) we conclude that $p \in \mathbb{S}^2(\mathbb{R})$ under $\mathbb{P}$.

We remark that deriving Theorem 5.1 is important from the point of view of possible applications, as finding a representation from Theorem 5.1 may lead to an explicit solution, in contrast to numerical methods applied to the fixed point equation (4.3) or the backward stochastic differential equation (4.2).

6 Equity-linked payment processes with independent insurance risk

In this section, under the conditions **(A1)-(A12)**, **(B1)-(B3)**, **(C1)-(C3)** we deal with the life and non-life payment processes (2.8)-(2.9), (2.10)-(2.11) and for these two important cases we derive solutions of the corresponding BSDEs (4.2) which allow us to express the optimal investment strategies in explicit form. In particular, we find explicit representations of the corresponding martingales from Theorem 5.1. We end up with a numerical example.

6.1 Explicit optimal strategies for special cases of the liability process

Let the target ψ consist of a reserve R for the outstanding liabilities and a profit requirement ϕ, so that the assumption **(D1)**, explained in Section 3, holds. An example of ϕ could be $\phi(t) = x_0(e^{ct} - 1), c > 0$. We adopt a market-consistent valuation basis and define the reserve, both in the case of the life and non-life insurance payment process, as the conditional expected value of future claims under an equivalent martingale measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ generated by (W, M, B, N). We assume for simplicity that the measure $\tilde{\mathbb{P}}$, generated by the combined financial and insurance market, coincides with the measure $\mathbb{Q}$, defined in (5.3), used for technical reasons to transform the equation (4.2). Let us remark that applying the measure $\mathbb{Q}$ as a pricing measure has a sound financial justification, as it is a so-called Föllmer-Schweizer minimal martingale measure which is commonly applied when dealing with pricing and hedging in incomplete markets, see Chapter 5.4.5 in Applebaum (2004). We point out that the case when the measures $\mathbb{Q}$ and $\tilde{\mathbb{P}}$ differ can still be handled but the calculations are more tedious.

In addition to **(B1)-(B3)**, introduced in Section 2.2.1 in the case of the life insurance payment process, we assume that

(B4) h, f, g satisfy the polynomial growth condition of order $k \in \mathbb{N}$ in s, i.e.

$$|h(t, s)| \leq K(1 + |s|^k), \quad (t, s) \in [0, T] \times (0, \infty),$$

and analogously for f, g.

In addition to **(C1)-(C3)**, introduced in Section 2.2.2 in the case of the non-life insurance payment process, we assume that

(C4) g_0 satisfies the polynomial growth condition of order $k \in \mathbb{N}$ in s,

$$|g_0(t, s, y)| \leq K(y)(1 + |s|^k), \quad (t, s, y) \in [0, T] \times (0, \infty) \times \mathbb{R},$$

with $\int_{\mathbb{R}} |K(y)|^2 q(dy) < \infty$.

The next assumptions concern both payment processes together. We assume that the moment conditions are finite:

(D2) $\sup_{t\in[0,T]} \mathbb{E}[|L(t)|^{2k}] < \infty$, which is equivalent to $\int_{\mathbb{A}} z^{2k}\nu(dz) < \infty$, $k \in \mathbb{N}$,

(D3) Λ is a strictly positive process and $\sup_{t\in[0,T]} \mathbb{E}\big[|\lambda(t)|^{2k}\big] < \infty$, $k \in \mathbb{N}$,

where Λ denotes, depending on the case, the underlying (background) intensity for the life-insurance or non-life insurance claims. The reader can check that **(B1)-(B4)**,**(C1)-(C4)** imply **(A6)-(A8)**. In practice, the functions h, f, g, g_0 should satisfy the growth conditions **(B4)**,**(C4)**. Indeed, the most common pay-offs: call options, put options, barrier options fulfill the requirements with $k = 1$. We define product measurable functions $\bar{h}, \bar{f}, \bar{g}, m, w, \bar{w} : [0, T] \times [0, T] \times (0, \infty) \to \mathbb{R}$,

$\bar{g}_0 : [0,T] \times [0,T] \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that for $0 \le t \le u \le T$,

$$\begin{aligned}
\bar{h}(t,u,s) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(u-t)}h(u,S(u))|S(t)=s\big],\\
\bar{f}(t,u,s) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(u-t)}f(u,S(u))|S(t)=s\big],\\
\bar{g}(t,u,s) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(u-t)}g(u,S(u))|S(t)=s\big],\\
\bar{g}_0(t,u,s,y) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(u-t)}g_0(u,S(u),y)|S(t)=s\big],\\
m(t,u,\lambda) &= \mathbb{E}^{\mathbb{Q}}\big[\lambda(u)|\lambda(t)=\lambda\big],\\
w(t,u,\lambda) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^u \lambda(v)dv}|\lambda(t)=\lambda\big],\\
\bar{w}(t,u,\lambda) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^u \lambda(v)dv}\lambda(u)|\lambda(t)=\lambda\big],
\end{aligned} \tag{6.1}$$

and for $0 \le u < t \le T$ we set $\bar{h} = \bar{f} = \bar{g} = \bar{g}_0 = m = w = \bar{w} = 0$. It can be shown that the conditional expected values are well defined. As Z and Λ are independent under $\mathbb{P}$, the last three expectations in (6.1) can be taken under $\mathbb{P}$ as well. Similarly to **(B4)**/**(C4)** , we require that

(D4) m satisfies the polynomial growth condition of order $k \in \mathbb{N}$ in λ,

which implies the same growth condition for $w, \bar{w}$ in λ. Moreover, one can show that the functions $\bar{h}, \bar{f}, \bar{g}, \bar{g}_0$ also inherit the growth condition in s from h, f, g, g_0; a simple proof is left to the reader. We remark that the growth condition **(D4)** on the expected intensity is very likely to hold. This condition is satisfied for example in the case of affine processes, which have been applied with success in mortality modelling, see Dahl et al. (2007), Dahl, Møller (2006). Finally, we assume that

(D5) for all $0 \le u \le T, y \in \mathbb{R}$, the functions $\bar{h}(.,u,.), \bar{f}(.,u,.), \bar{g}(.,u,.), \bar{g}_0(.,u,.,y), \bar{m}(.,u,.), w(.,u,.), \bar{w}(.,u,.) : [0,u] \times (0,\infty) \to \mathbb{R}$ are continuously differentiable in the class $\mathcal{C}^{1,2}(\mathbb{R})$ and product measurable in (t,u,s,y).

The assumption **(D5)** is mathematically convenient, as it allows us to find a martingale representation in a rather straightforward way. Otherwise, we would have to use Malliavin calculus as in Ankirchner, Imkeller (2008). This requirement should also be fulfilled in practice. We recall that the prices of call options, put options, barrier options in the Lévy-diffusion model and survival probabilities for affine processes are sufficiently smooth under mild conditions, see Chapter 12.1 in Cont, Tankov (2004) and Appendix in Cairns (2004).

Let us now investigate the reserve R which, from its definition introduced at the beginning of this section, is calculated as the expected value

$$R(t) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-r(s-t)}dP(s)\big|\mathcal{F}_t\Big], \quad 0 \le t \le T. \tag{6.2}$$

In the case of the life insurance payment process P defined in (2.8), we can represent the reserve, for $0 \le t \le T$, as

$$R^{life}(t) = (n - Y(t))w(t,T,\lambda(t))\bar{f}(t,T,S(t)) \\ +(n-Y(t))\int_t^T \Big(w(t,s,\lambda(t))\bar{h}(t,s,S(t)) + \bar{w}(t,s,\lambda(t))\bar{g}(t,s,S(t))\Big)ds, \quad (6.3)$$

where, when calculating the expected value, we apply the independence assumptions **(B1)-(B2)**, holding under $\mathbb{Q}$ as well, the definitions (6.1), the martingale property of the compensated integral driven by $\tilde{N}$ and the relation

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\big[n - Y(s)|\mathcal{F}_t\big] &= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\big[n - Y(s)|\mathcal{F}_t \vee \sigma(\lambda(u), u \le T)\big]\Big|\mathcal{F}_t\Big] \\ &= (n - Y(t))\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^s \lambda(u)du}|\mathcal{F}_t\big], \quad 0 \le t \le s \le T. \quad (6.4) \end{aligned}$$

The equality (6.4) holds by the properties of conditional expectations and, conditional on $\sigma(\lambda(s), s \le T)$, Bernoulli distribution of the number of survivors. Notice that the Bernoulli distribution can be deduced from the independence assumption of the lifetimes stated in **(B2)**, which holds as well under $\mathbb{Q}$ due to the unchanged mortality dynamics.

Similarly, in the case of the non-life insurance payment process P defined in (2.10), we can represent the reserve, for $0 \le t \le T$, as

$$R^{nonlife}(t) = \int_t^T \int_{\mathbb{R}} \bar{g}_0(t,s,S(t),y)m(t,s,\lambda(s))q(dy)ds. \quad (6.5)$$

The assumptions made in this section imply that $R : \Omega \times [0,T] \to \mathbb{R}$ has càdlàg sample paths and there exists a product measurable function $R(t,s,\lambda,y)$ such that $R(t,S(t),\lambda(t),Y(t)) = R(t)$, $\mathbb{Q}$-a.s., $0 \le t \le T$. Moreover, R inherits the growth condition $|R(t,s,\lambda,y)| \le K(1 + |s\lambda|^k)$. Hence by independence of S and Λ the reserve is square integrable under $\mathbb{P}$ and the condition **(A10)** is fulfilled.

We now give the explicit expressions for (β,κ), which are the key components of the solution to the BSDE (4.2) required to implement the optimal strategy (4.4).

Lemma 6.1. *Consider the life insurance payment process* (2.8)-(2.9). *Assume that* ***(A1)-(A5), (A12), (B1)-(B4), (D1)-(D5)*** *hold, the profit target ϕ is a continuous function $t \mapsto \phi(t)$ and the reserve R is calculated as in* (6.3). *Define*

$$\tilde{A}(s,t) = -2e^{(\delta+r)(s-t)}\tilde{p}(s) + \theta\frac{e^{(\delta+r)(s-t)} - 1}{\delta + r}, \quad 0 \le t \le s \le T.$$

The corresponding solution (β,κ) of the BSDE (4.2) *is given by*

$$\beta(t) = \tilde{A}(T,t)(n - Y(t-))S(t-)\sigma w(t,T,\lambda(t))\bar{f}_s(t,T,S(t-)) \\ +(n-Y(t-))S(t-)\sigma \\ \times \int_t^T \tilde{A}(s,t)\Big(w(t,s,\lambda(t))\bar{h}_s(t,s,S(t-)) + \bar{w}(t,s,\lambda(t))\bar{g}_s(t,s,S(t-))\Big)ds,$$

$$\kappa(t,z) = \tilde{A}(T,t)(n-Y(t-))w(t,T,\lambda(t))\Big(\bar{f}(t,T,S(t-)(1+z)) - \bar{f}(t,T,S(t-))\Big)$$
$$+(n-Y(t-))\int_t^T \tilde{A}(s,t)\Big(w(t,s,\lambda(t))\big(\bar{h}(t,s,S(t-)(1+z)) - \bar{h}(t,s,S(t-))\big)$$
$$+\bar{w}(t,s,\lambda(t))\big(\bar{g}(t,s,S(t-)(1+z)) - \bar{g}(t,s,S(t-))\big)\Big)ds.$$

Lemma 6.2. *Consider the non-life insurance payment process* (2.10)-(2.11). *Assume that **(A1)-(A5)**, **(A12)**, **(C1)-(C4)**, **(D1)-(D5)** hold, the profit target ϕ is a continuous function $t \mapsto \phi(t)$ and the reserve R is calculated as in* (6.5). *Define*

$$\tilde{A}(s,t) = -2e^{(\delta+r)(s-t)}\tilde{p}(s) + \theta\frac{e^{(\delta+r)(s-t)}-1}{\delta+r}, \quad 0 \le t \le s \le T.$$

The corresponding solution (β,κ) of the BSDE (4.2) *is given by*

$$\beta(t) = S(t-)\sigma\int_t^T \tilde{A}(s,t)m(t,s,\lambda(t))\int_{\mathbb{R}} \bar{g}_{0,s}(t,s,S(t-),y)q(dy)ds,$$

$$\kappa(t,z) = \int_t^T \tilde{A}(s,t)m(t,s,\lambda(t))$$
$$\times\int_{\mathbb{R}} \big(\bar{g}(t,s,S(t-)(1+z),y) - \bar{g}(t,s,S(t-),y)\big)q(dy)ds.$$

We remark that the subscript s denotes the first derivative with respect to the variable s. The above results follow by establishing the explicit martingale representation based on Theorem 5.1. The details of the derivations are presented in the Appendix.

Let us comment on the structure of the optimal investment strategy. By investigating the above explicit expressions for (β,κ) we can conclude that the optimal investment strategy (4.4) is a kind of extended delta-hedging strategy (compare with Chapter 10.4.2 in Cont, Tankov (2004)) as it contains terms which hedge continuous and discontinuous changes in the risky-asset's price, weighted by some probabilities related to insurance factors which are not tradeable in the market. The processes (β,κ) can be interpreted as these are the amounts needed to hedge continuous and discontinuous changes in the risky-asset's price at the times when a death benefit (or a non-life claim), a survival benefit or a terminal benefit is paid in the future.

6.2 A numerical example

In this subsection a simple numerical example is considered to illustrate the results obtained in this paper. We investigate the non-life insurance payment process (2.10)-(2.11). We assume that the insurance claims are generated by a compound Cox process Y, or the corresponding random measure N. The compound Cox process consists of an independent Cox process with a stochastic intensity Λ driven by the exponential martingale

$$\lambda(t) = 10e^{-0.08t+0.4B(t)}, \quad 0 \le t \le T,$$

and the claims' sizes $(V_i)_{i\in\mathbb{N}}$ which are independent and identically exponentially distributed with expected value 60. The price of the risky asset S follows the exponential jump-diffusion (Merton model)

$$S(t) = 100e^{-0.1331t+0.2W(t)+\sum_{i=1}^{M(t)} Z_i}, \quad 0 \le t \le T,$$

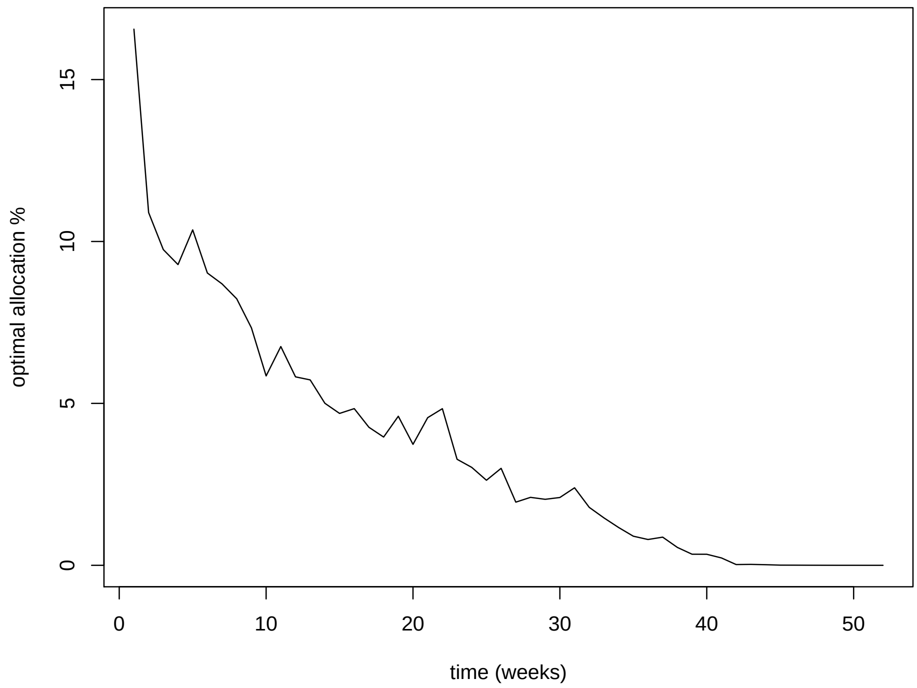
where $(M(t), 0 \le t \le T)$ denotes an independent Poisson process with intensity 10 and $(Z_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically normally distributed random variables $Z_i \sim \mathcal{N}(0, 0.15^2)$. We remark that similar forms for Λ have been proposed in life insurance to model stochastic mortality, see Ballotta, Haberman (2006), and that the Merton model is the classical example of a Lévy driven stock model, see Chapter 4.3 in Cont, Tankov (2004). We assume that Y and S are independent as well. We set $r = \mu = 0$, so that the price process S is already a martingale under the real world measure $\mathbb{P}$; hence the value of the drift, equal to -0.1331, in the exponent for S. The insurer's initial capital is taken to be $x_0 = 600$. We deal with a stop-loss contract with a barrier, see Møller (2003), with duration of 1 year, $T = 1$, under which the insurer covers claims $V_1, V_2, ...,$ over the threshold of 30 provided that the tradeable asset's value S at the time of the claim is over 100. Our payment process takes the form

$$\begin{aligned} P(t) &= \sum_{u\in(0,t]} \mathbf{1}\{S(u) > 100\}(\triangle Y(u) - 30)^+ \mathbf{1}_{\{\triangle Y(u)\neq 0\}}(u) \\ &= \int_0^t \int_{\mathbb{R}} \mathbf{1}\{S(u-) > 100\}(y - 30)^+ N(du, dy), \quad 0 \le t \le T, \end{aligned}$$

and fits into the framework of equity-linked payment processes with independent unsystematic and systematic insurance loss risk and Lemma 6.2. We consider only the terminal penalty, $\theta = 0$, and as we work under a martingale measure the value of $\phi(T)$ does not matter, as an investment strategy only controls deviations of the wealth and not its drift. We remark that as long as the Lévy process L_E has a diffusion component, the function $(t, s) \mapsto \bar{g}_0(t, u, s) = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{S(u) > 100\}|S(t) = s]$ is sufficiently smooth, see Proposition 12.5 in Cont, Tankov (2004).

We point out that we only aim at illustrating the performance of the derived optimal strategy. We remark that all parameters in this example have been chosen arbitrarily and have no relations to real market data. We realize that some of the assumptions made (mainly for the purpose of quick calculations) might be overly simplistic and would not be made during a real asset-liability management exercise.

Figure 1: An example of the optimal allocation in % of wealth over the duration of the contract.



Some more worthwhile results and conclusions might be obtained by performing an extensive simulation study.

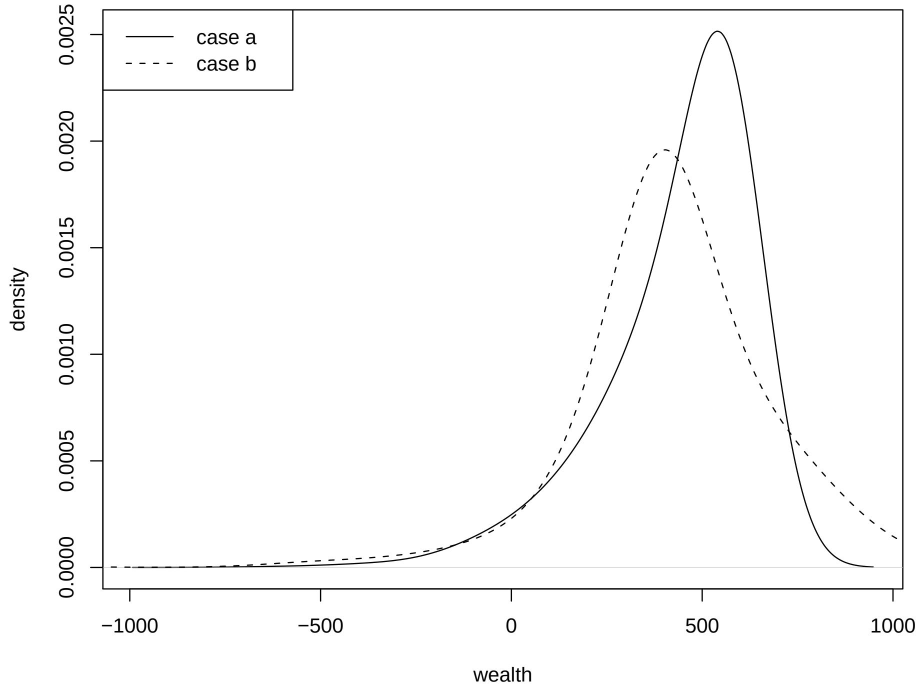
Figure 1 presents an example of an evolution, over the duration of a contract, of the optimal allocation calculated according to Theorem 4.1. The optimal allocation, measured in % of available wealth, ranges from 17% at the inception of the contract and decreases gradually to 0% at the end of the contract. Such decreasing dynamics is typical of quadratic hedging strategies. The goal of the optimization is to stabilize deviations of the terminal wealth and intuitively this can be achieved by investing more in the risky asset at the beginning and less towards the end, as excessive holdings of the risky asset increase deviations of the terminal wealth. Even though the path of the allocation dynamics is not surprising, the correct application depends on our Lemma 6.2 and without this result it would not be possible.

Let us now consider two cases. In case a) the insurer invests according to the optimal strategy (4.4), and in case b) the insurer applies the classical, and most common, delta-hedging strategy

$$\tilde{\pi}(t) = S(t-)\lambda(0) \int_t^T \bar{g}_{0,s}(t, s, S(t-))ds \int_{30}^{\infty} (v - 30)\frac{1}{60}e^{-\frac{1}{60}v}dv, \quad 0 \leq t \leq T, \tag{6.6}$$

which does not take into account the randomness of Λ and jumps in the risky asset's dynamics. We are interested in consequences of omitting these risk factors or

Figure 2: Empirical densities of the terminal wealths at the end of a contract.



applying the strategy which is not fully calibrated to the underlying risk process. The strategy considered in case a) can be seen as a tailor-made strategy, optimal in the given specific risk model, and depending on the set of incorporated risk factors. On the contrary, the strategy considered in case b) can be seen as a general strategy, which behaves and evolves according to our expectations and common knowledge, but whose optimality holds only in a simplistic Black-Scholes model incorporating a limited number of risk factors. Figure 2 presents the empirical density functions of the terminal wealth arising in cases a) and b). As we work under a martingale measure, the expected terminal wealths coincide in both cases and equal around 442. However, the deviations and percentiles differ significantly. The density in case a) is more centrally concentrated compared to case b) which is more diffuse; see also Table 1. The deviation of the terminal wealth in case a) equals 182.84 compared to the deviation 281.33 in case b). Moreover, the probability that the controlled wealth drops below zero equals 3.32% in case a) and 4.95% in case b). In our opinion the differences are significant and the risk profile of the company improves with the tailor-made strategy. The advantage of applying the derived optimal strategy (4.4) compared to the classical delta-hedging strategy (6.6) becomes the more apparent, the larger the variability of Λ or intensity/variability of jumps of the risky asset is. In our opinion this simple numerical example confirms that applying a more sophisticated, risk/model-specific, but numerically demanding, strategy (4.4) is worthwhile,

Table 1: Percentiles of the terminal wealth at the end of the contract

	Case "a"	Case "b"
1st percentile	-168.95	-516.00
5th percentile	86.64	4.24
95th percentile	590.74	809.22
99th percentile	608.76	992.15

because it puts the insurer in a safer and more stable financial position. Our conclusion coincides with the common knowledge among theoreticians and practitioners that risk-specific/company-specific internal models are a must, even if this means more demanding calculations. If we can quantify the particular risk factors which affect the performance of an insurance portfolio, we should use the estimates in the risk analysis. Neglecting risk factors may have far reaching consequences.

7 Conclusions

In this paper we have investigated an asset-liability management problem for a stream of liabilities, generated by a step process (or a random measure) with a stochastic intensity, written on liquid traded assets and non-traded sources of risks in a financial market driven by a Lévy process. The framework is very general and allows one to investigate index-linked, path-dependent, liabilities under systematic and unsystematic insurance risk and financial risk simultaneously. We have derived the optimal investment strategy in the model considered. Applications of the strategy relies on finding a representation of a martingale or solving a linear backward stochastic differential equation. For the most common payment processes arising in life and non-life insurance businesses we have found explicit martingale representations. In other cases we must resort to numerical schemes for BSDEs. We believe that our results contribute both to the practical area of asset-liability management or hedging and to the theory of optimal control.

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8 The appendix: additional results and proofs

Let $L^2_\nu(\mathbb{R})$ and $L^2_{\xi Q}(\mathbb{R})$ denote (respectively) the spaces of measurable functions $\kappa : \mathbb{R} \to \mathbb{R}$ and $\eta : \mathbb{R} \to \mathbb{R}$ satisfying the integrability requirements

$$\begin{aligned} &\textstyle\int_{\mathbb{R}} |\kappa(z)|^2 \nu(dz) < \infty, \\ &\textstyle\int_{\mathbb{R}} |\eta(y)|^2 \xi(t) Q(t, dy) < \infty, \quad \text{a.e.} t \in [0,T], \mathbb{P}-\text{a.s.}. \end{aligned}$$

We define $\alpha : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\xi Q}(\mathbb{R}) \to \mathbb{R}$ to be the generator of the BSDE (4.2), which is linear in the sense that there exists a constant K such that for a.e. $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned} |\alpha(t, p_1, \beta_1, \gamma_1, \kappa_1, \eta_1) - \alpha(t, p_2, \beta_2, \gamma_2, \kappa_2, \eta_2)|^2 \leq K\Big(|p_1 - p_2|^2 + |\gamma_1 - \gamma_2|^2 \\ + \textstyle\int_{\mathbb{R}} |\kappa_1(z) - \kappa_2(z)|^2 \nu(dz) + \int_{\mathbb{R}} |\eta_1(y) - \eta_2(y)|^2 \xi(t) Q(t, dy)\Big), \end{aligned}$$

for any $(p_1, \beta_1, \gamma_1, \kappa_1, \eta_1) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\xi Q}(\mathbb{R})$, $(p_2, \beta_2, \gamma_2, \kappa_2, \eta_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\xi Q}(\mathbb{R})$. The equation (4.2) is a linear BSDE driven by Brownian motions and random measures.

Backward stochastic differential equations driven by Brownian motions and random measures with Lipschitz continuous generators have been considered in Becherer (2006). Our equation (4.2) does not fit exactly into the framework of Proposition 3.2 from Becherer (2006), as our jump intensity ξ of Y is not bounded. However, due to the second square integrability assumption in **(A7)** the required extension is possible.

Lemma 8.1. *Assume that **(A1)-(A11)** hold. There exists a unique solution* $(p, \beta, \gamma, \kappa, \eta) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_M(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ *of* (4.2).

Proof:
The proof is standard and, as always, relies on the representation property **(A11)** and a contraction inequality in appropriate Banach spaces from Definition 3.2. The reader can consult Theorem 2.1 in El Karoui et al. (1997) or Proposition 3.2 in Becherer (2006). The details of the proof can be obtained from the author upon request. We only notice that the terminal condition $p(T) = -2\psi(T)$ is a square integrable random variable and

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T |\alpha(s,0,0,0,0,0)|^2 ds\Big] \\ &\leq \; K\mathbb{E}\Big[\int_0^T |\psi(s)|^2 ds + \int_0^T |H(s)|^2 ds + \int_0^T \int_{\mathbb{R}} |G(s,y)\xi(s)|^2 Q(s,dy) ds\Big] < \infty, \end{aligned}$$

due to our assumptions **(A6)**, **(A7)**, **(A10)**, and we can follow the idea of the proofs from the above cited references. □

Lemma 8.2. *Under **(A1)-(A11)**, the investment strategy* (4.4) *is admissible,* $\pi \in \mathcal{A}$.

Proof:
For simplicity let us denote $\pi(t) = AX^{\pi}(t-) + B(t)$. As p is a càdlàg mapping, the process $p(t-)$ is left-continuous with right-limits, hence predictable. By predictability of β, κ, established in Lemma 8.1, we conclude that B is predictable as well. Moreover, by Cauchy-Schwarz' inequality and square integrability of β, κ, established in Lemma 8.1, we have

$$\begin{aligned}
&\mathbb{E}\big[\int_0^T |B(s)|^2 ds\big] \\
&\leq K\big(\mathbb{E}\big[\int_0^T |p(s)|^2 ds + \int_0^T |\beta(s)|^2 ds + \int_0^T\int_{\mathbb{R}} |\kappa(s,z)|^2 \nu(dz)ds\big]\big) < \infty.
\end{aligned}$$

The dynamics of the wealth process (3.1) under the strategy (4.4) is given by

$$\begin{aligned}
dX^{\pi}(t) &= X^{\pi}(t-)\big(\big(r + (\mu - r)A\big)dt + A\sigma dW(t) + A\int_{\mathbb{R}} z\tilde{M}(dt,dz)\big) \\
&\quad + B(t)\big((\mu - r)dt + \sigma dW(t) + \int_{\mathbb{R}} z\tilde{M}(dt,dz)\big) \\
&\quad - H(t)dt - \int_{\mathbb{R}} G(t,y)N(dt,dy), \quad 0 \leq t \leq T.
\end{aligned}$$

The above stochastic differential equation is a Lipschitz linear equation driven by a semimartingale (a Lévy process) with a càdlàg drift and there exists a unique càdlàg solution X^{π} on $[0,T]$, see Theorem V.7 in Protter (2004). Moreover, $X^{\pi}(t-)$ is left-continuous with right limits. We conclude that π is a predictable process.
Define a sequence of stopping times $\tau_n = \inf\{t \geq 0, |X^{\pi}(t)| \geq n\} \wedge T$. By Cauchy-Schwarz' inequality, square integrability of (localized) stochastic integrals, square integrability of B and assumptions **(A6)**- **(A7)** we derive

$$\begin{aligned}
&\mathbb{E}\big[|X^{\pi}(t)|^2 \mathbf{1}\{t \leq \tau_n\}\big] \leq \mathbb{E}\big[|X^{\pi}(t\wedge\tau_n)|^2\big] \leq K\Big(1 + \mathbb{E}\big[\int_0^t |X^{\pi}(s-)|^2 \mathbf{1}\{s \leq \tau_n\} ds\big] \\
&\quad + \mathbb{E}\big[\int_0^T |B(s)|^2 ds + \int_0^T |H(s)|^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}} |G(s,y)|^2 \xi(s) Q(s,dy)ds + \int_0^T\int_{\mathbb{R}} |G(s,y)\xi(s)|^2 Q(s,dy)ds\big]\Big) \\
&\leq K\big(1 + \mathbb{E}\big[\int_0^t |X^{\pi}(s)|^2 \mathbf{1}\{s \leq \tau_n\}\big]ds\big), \quad 0 \leq t \leq T,
\end{aligned}$$

by Gronwall's inequality we have

$$\mathbb{E}\big[|X^{\pi}(t)|^2 \mathbf{1}\{t \leq \tau_n\}\big] \leq Ke^{KT}, \quad 0 \leq t \leq T,$$

and letting $n \to \infty$, together with Fatou's lemma, yields

$$\sup_{t\in[0,T]} \mathbb{E}\big[|X^\pi(t)|^2\big] < \infty. \tag{8.1}$$

This, together with square integrability of B, gives the required square integrability of π and the admissibility of the strategy (4.4) is proved. □

The proof of Lemma 6.1.:

We only give the proof for the life insurance payment process, as the calculations for the non-life insurance payment process are analogous and less demanding.

Let $0 \le t \le T$. The weak property of predictable representation **(A11)** holds as the payment process P is driven by a random measure generated by a step process and we shall work with the natural filtration. As **(A12)** is assumed to hold we apply Theorem 5.1 and we have to find a representation of the uniformly integrable martingale

$$\mathbb{E}^{\mathbb{Q}}\Big[-2e^{\delta T}\Big(\phi(T)+(n-Y(T))f(T,S(T))\Big)-2\int_0^T e^{\delta s}\Big(\theta\phi(s)+\theta R(s) \\ +\tilde{p}(s)\Big((n-Y(s))h(s,S(s))+g(s,S(s-))(n-Y(s-))\lambda(s)\Big)\Big)ds\Big|\mathcal{F}_t\Big]. \tag{8.2}$$

We show that there exists a representation of (8.2) consisting of a constant and stochastic integrals driven by $W^{\mathbb{Q}}, B, \tilde{M}^{\mathbb{Q}}, \tilde{N}$ in the sense of local martingales. When deriving a representation we omit Lebesgue integrals, as being continuous martingales with paths of finite variation on compacts they are constant. We only collect stochastic Itô integrals, local martingales driven by $W^{\mathbb{Q}}, B, \tilde{M}^{\mathbb{Q}}, \tilde{N}$ which form the martingales

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{\delta T}(n-Y(T))f(T,S(T))|\mathcal{F}_t\Big], \\ \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{\delta s}\Big(\theta R(s)+\tilde{p}(s)(n-Y(s))\Big(h(s,S(s))+g(s,S(s))\lambda(s)\Big)\Big)ds\Big|\mathcal{F}_t\Big] \tag{8.3}$$

where we have already omitted the Lebesgue integral from 0 to t, together with a constant -2 and deterministic ϕ. We remark that in finding a representation of a martingale, closed by a random variable, we follow the ideas from Appendix in Møller (2001).

First, by (6.4) and independence of (Y,Λ) and S assumed in **(B1)**-**(B2)** we calculate the conditional expected values

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{\delta T}(n-Y(T))f(t,S(T))|\mathcal{F}_t\Big] = e^{\delta T}(n-Y(t))w(t,T,\lambda(t))e^{r(T-t)}\bar{f}(t,T,S(t)), \tag{8.4}$$

and

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{\delta s}\tilde{p}(s)(n-Y(s))\Big(h(s,S(s))+g(s,S(s))\lambda(s)\Big)ds\big|\mathcal{F}_t\Big] = (n-Y(t)) \times \int_t^T e^{\delta s+r(s-t)}\tilde{p}(s)\Big(w(t,s,\lambda(t))\bar{h}(t,s,S(t))+\bar{w}(t,s,\lambda(t))\bar{g}(t,s,S(t))\Big)ds. \tag{8.5}$$

By the properties of conditional expectations and a similar conditioning argument to the one in (6.4) we derive that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\big[e^{-r(s-t)}\bar{f}(s,u,S(s)))|\mathcal{F}_t\big] = \bar{f}(t,u,S(t))), \quad 0\le t\le s\le u\le T,\\
&\mathbb{E}^{\mathbb{Q}}\big[(n-Y(s))w(s,u,\lambda(s))|\mathcal{F}_t\big] = (n-Y(t))w(t,u,\lambda(t)), \quad 0\le t\le s\le u\le T,\\
&\mathbb{E}^{\mathbb{Q}}\big[(n-Y(s))\bar{w}(s,u,\lambda(s))|\mathcal{F}_t\big] = (n-Y(t))\bar{w}(t,u,\lambda(t)), \quad 0\le t\le s\le u\le T,
\end{aligned}$$

hold $\mathbb{Q}$-a.s., which together with the representation of the reserve (6.3) and Fubini's theorem yields that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{\delta s}R(s)ds\Big|\mathcal{F}_t\Big] &= (n-Y(t))w(t,T,\lambda(t))\bar{f}(t,T,S(t))\int_t^T e^{\delta s+r(s-t)}ds\\
&+(n-Y(t))\\
&\times\int_t^T e^{\delta s+r(s-t)}\int_s^T\Big(w(t,u,\lambda(t))\bar{h}(t,u,S(t))+\bar{w}(t,u,\lambda(t))\bar{g}(t,u,S(t))\Big)duds.
\end{aligned}$$

Finally by changing the order of integration we arrive at

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{\delta s}R(s)ds\Big|\mathcal{F}_t\Big]\\
&= (n-Y(t))w(t,T,\lambda(t))\bar{f}(t,T,S(t))e^{-rt}\frac{e^{(\delta+r)T}-e^{(\delta+r)t}}{\delta+r}+(n-Y(t))\\
&\quad\times\int_t^T e^{-rt}\frac{e^{(\delta+r)u}-e^{(\delta+r)t}}{\delta+r}\Big(w(t,u,\lambda(t))\bar{h}(t,u,S(t))+\bar{w}(t,u,\lambda(t))\bar{g}(t,u,S(t))\Big)du.
\end{aligned} \tag{8.6}$$

As (8.4), (8.5) and (8.6) are the building blocks (8.3), instead of dealing with (8.3) we investigate the processes

$$A(T,t)(n-Y(t))w(t,T,\lambda(t))\bar{f}(t,T,S(t)), \tag{8.7}$$

$$\int_t^T A(s,t)(n-Y(t))\Big(w(t,s,\lambda(t))\bar{h}(t,s,S(t))+\bar{w}(t,s,\lambda(t))\bar{g}(t,s,S(t))\Big)ds, \tag{8.8}$$

with

$$A(s,t) = e^{\delta s}e^{r(s-t)}\tilde{p}(s)+\theta e^{-rt}\frac{e^{(\delta+r)s}-e^{(\delta+r)t}}{\delta+r}, \quad 0\le t\le s\le T.$$

By the assumption **(D5)**, we can apply the Itô formula to the product (8.7). Collecting the stochastic integrals driven by $W^{\mathbb{Q}}, \tilde{M}^{\mathbb{Q}}$, as only these terms are required to derive the optimal investment strategy (4.4), we obtain

$$\begin{aligned}
&\int_0^t A(T,s)(n-Y(s-))w(s,T,\lambda(s))\bar{f}_s(s,T,S(s-))\sigma S(s-)dW^{\mathbb{Q}}(s),\\
&\int_0^t\int_{\mathbb{R}} A(T,s)(n-Y(s-))w(s,T,\lambda(s))\\
&\qquad\times\Big(\bar{f}(s,T,S(s-)(1+z))-\bar{f}(s,T,S(s-))\Big)\tilde{M}^{\mathbb{Q}}(ds,dz).
\end{aligned}\tag{8.9}$$

It is easy to see that the stochastic integral with respect to $W^{\mathbb{Q}}$ is a local martingale. The second integral can be decomposed into

$$\begin{aligned}
&\int_0^t\int_{|z|\le 1} A(T,s)(n-Y(s-))w(s,T,\lambda(s))\\
&\qquad\times\Big(\bar{f}(s,T,S(s-)(1+z))-\bar{f}(s,T,S(s-))\Big)\tilde{M}^{\mathbb{Q}}(ds,dz),\\
&\int_0^t\int_{z>1} A(T,s)(n-Y(s-))w(s,T,\lambda(s))\\
&\qquad\times\Big(\bar{f}(s,T,S(s-)(1+z))-\bar{f}(s,T,S(s-))\Big)\tilde{M}^{\mathbb{Q}}(ds,dz),
\end{aligned}$$

and we observe that the first term is a local martingale by the mean-value theorem, and the second term is a local martingale by the growth condition following from **(B4)** together with the integrability assumption **(D2)**. We also apply the Itô formula to the product under the integral in (8.8) and arrive at

$$\begin{aligned}
&\int_0^t A(s,u)(n-Y(u-))w(u,s,\lambda(u))\bar{h}_s(u,s,S(u-))\sigma S(u-)dW^{\mathbb{Q}}(u),\\
&\int_0^t\int_{\mathbb{R}} A(s,u)(n-Y(u-))w(u,s,\lambda(u))\\
&\qquad\times\Big(\bar{f}(u,s,S(u-)(1+z))-\bar{f}(u,s,S(u-))\Big)\tilde{M}^{\mathbb{Q}}(du,dz),
\end{aligned}\tag{8.10}$$

and analogously for the product $A(n-Y)\bar{w}\bar{g}$. By Fubini's theorem for stochastic integrals, see Theorem IV.65 in Protter (2004) and Appendix in Møller (2001) for more details, we can change the order of integrations in (8.8), (8.10) and obtain the required terms

$$\begin{aligned}
&\int_0^t (n-Y(u-))\int_u^T A(s,u)\Big(w(u,s,\lambda(u))\bar{h}_s(u,s,S(u-))\\
&\quad+\bar{w}(u,s,\lambda(u))\bar{g}_s(u,s,S(u-))\Big)ds\sigma S(u-)dW^{\mathbb{Q}}(u),\\
&\int_0^t\int_{\mathbb{R}}(n-Y(u-))\int_u^T A(s,u)\Big(w(u,s,\lambda(u))\big(\bar{h}(u,s,S(u-)(1+z))-\bar{h}(u,s,S(u-))\big)\\
&\quad+\bar{w}(u,s,\lambda(u))\big(\bar{g}(u,s,S(u-)(1+z))-\bar{g}(u,s,S(u-))\big)\Big)ds\tilde{M}^{\mathbb{Q}}(du,dz).
\end{aligned}\tag{8.11}$$

The stochastic integrals in (8.9) and (8.11) provide us (after multiplying by -2) with candidates for $(\bar{\beta}, \bar{\kappa})$. In the same way we can also recover candidates for $(\bar{\gamma}, \bar{\eta})$. To conclude we have to prove that the derived candidates $(\bar{\beta}, \bar{\gamma}, \bar{\kappa}, \bar{\eta})$ are in the class $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_M(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ under the measure $\mathbb{P}$, as required in Theorem 5.1. We claim that the candidate solution satisfies the integrability requirements and indeed represents the uniformly integrable martingale (8.2). The details are omitted but the proof of this claim, and the justification for applying Fubini's lemma for stochastic integrals, can be obtained from the author upon request. A solution $(\beta, \gamma, \kappa, \eta)$ of (4.2) is obtained by multiplying $(\bar{\beta}, \bar{\gamma}, \bar{\kappa}, \bar{\eta})$ by $e^{-\delta t}$. The proof is now complete.

Notice that the solution p, defined in Theorem 5.1, can be derived explicitly from (8.3), (8.4) and (8.5). $\square$