

Pricing and Hedging of Variable Annuities with State-Dependent Fees

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Abstract

We investigate the problem of pricing and hedging variable annuity contracts for which the fee deducted from the policyholder's account depends on the account value. It is believed that state-dependent fees are beneficial to policyholders and insurers since they reduce policyholders' incentives to lapse the policies and match the costs incurred by policyholders with the pay-offs received from embedded guarantees. We consider an incomplete financial market which consists of two risky assets modelled with a two-dimensional Lévy process. One of the assets is a security which can be traded by the insurer, and the second asset is a security which is the underlying fund for the variable annuity contract. In our model we derive an equation from which the fee for the guaranteed benefit can be calculated and we characterize a strategy which allows the insurer to hedge the benefit. To solve the pricing and hedging problem in an incomplete financial market we apply a quadratic objective.

Keywords: Quadratic optimization, incomplete market, Lévy process, Backward Stochastic Differential Equations, Lévy Clayton copula.

JEL: C61, G11, G13.

1 Introduction

Variable annuities are among the most popular insurance contracts sold worldwide. Their popularity is due to the fact that variable annuities combine insurance with investment by providing a protection against life contingencies and a participation in the growth of the financial market. Variable annuities provide benefits which are contingent on the performance of investment funds together with capital protections which guarantee a minimum rate of return from the investment. Nowadays, we find a range of variable annuity contracts which guarantee a minimum death benefit, minimum maturity benefit, minimum income benefit and minimum accumulation benefit.

The problem of pricing and hedging variable annuities has been thoroughly studied in the actuarial literature, see among others Bacinello et. al. (2011), Bauer et. al. (2008), Bernard et. al. (2014), Coleman et. al. (2007), Deelstra and Rayée (2013), Hardy (2003), Quittard-Pinon and Kelani (2013). From the financial point of view the capital protection embedded in a variable annuity is a financial option on an investment fund. Consequently, techniques from financial mathematics should be applied in order to price and hedge variable annuity benefits. However, there is a significant difference between pricing and hedging financial options and guarantees embedded in variable annuities. A financial option is financed with a premium which is paid by the buyer of the option at the inception of the contract, whereas a guarantee embedded in a variable annuity is financed with fees which are paid by the policyholder during the lifetime of the contract. Moreover, the fees are deducted from the policyholder's account and those fees should finance the guarantee which is contingent on the policyholder's account value. Those subtle issues, typical for variable annuities, should be reflected in a model which is used for pricing and hedging of variable annuities.

In most variable annuity contracts insurers deduct fees which are proportional to the policyholder's account value. Consequently, if the account value is low the fee is low, and if the account value is high the fee is high. It has been noticed that such a fee payment scheme increases incentives among insured persons to lapse their policies.

Guarantees which are embedded in variable annuities are similar to put options, which means that the guarantee is in-the-money if the account value is low and is out-of-the-money if the account value is high. If a proportional fee is deducted from the account, then the policyholder pays a high fee for the guarantee in times when the guarantee is not valuable to him. Clearly, the policyholder is not satisfied if he has to pay a lot of money for the embedded guarantee which he does not need in times of growing economy and, consequently, he is very likely to lapse the policy. In order to reduce policyholders' incentives for lapsing variable annuities it has been suggested that state-dependent fees should be introduced by insurers. In recent years Prudential UK introduced a variable annuity with a guaranteed minimum return under which the fee is deducted from the account at a fixed rate only if the account value is below a guaranteed level. Under such a account-dependent payment scheme the fee for the guarantee is paid only if the guarantee is valuable to the policyholder. The advantage of such a fee payment scheme is that it reduces policyholders' incentives to lapse the policies and matches the costs incurred by policyholders with the pay-offs received from embedded guarantees, but the disadvantage is that the insurer who collects the fee only in times when the guarantee is in-the-money must set the fee rate at a level which is higher than the constant fee rate.

A variable annuity contract with a fee which is deducted at a fixed rate only if the account value is below a pre-specified level has been recently studied in Bernard et. al. (2014). The authors consider a complete Black-Scholes financial model with one risky asset and derive an equation from which the fee for the embedded guarantee can be calculated. The problem of hedging the guaranteed benefit is not considered in Bernard et. al. (2014). In fact, the hedging strategy in the model from Bernard et. al. (2014) is trivial since the authors consider a complete financial market and, consequently, the delta-hedging strategy (the replicating strategy) is the only hedging strategy which can be used. To the best of our knowledge the paper by Bernard et. al. (2014) is the only paper in the literature which studies variable annuities with state-dependent fees. Our paper is the second one in this field. We would like to point out that our financial

model and our pricing and hedging problem are more general than the model and the problem from Bernard et. al. (2014).

In this paper we consider an incomplete financial market which consists of two risky assets modelled with a two-dimensional Lévy process. One of the assets is a security which can be traded by the insurer, and the second asset is a security which is the underlying fund for the variable annuity contract. Hence, in this paper we take into account two important sources of market incompleteness which the insurer must face in reality. The first source of market incompleteness comes from unpredictable jumps (crashes) in the asset price which are modelled with a discontinuous Lévy process, and the second source of market incompleteness comes from the impossibility to trade the fund on which the variable annuity is contingent. We would like to point out that in reality the insurer can never trade the underlying fund (an exotic external fund) for a variable annuity and asymmetric heavy tails of asset returns and crashes in the market are the main financial risks for the insurer selling a variable annuity. As far as the fee payment scheme is concerned, which is the crucial point in our paper, we consider a general state-dependent fee which is modelled as a function of the account value. Our fee process includes the fee process considered in Bernard et. al. (2014). To solve the pricing and hedging problem in our incomplete financial model we apply a quadratic objective and we require that the mismatch between the hedging portfolio and the liability at the terminal time is minimal in a mean-square sense. We derive an equation from which the fee for the guaranteed benefit can be calculated and we find the hedging strategy which allows the insurer to hedge optimally the benefit. We use a backward stochastic differential equation to characterize the fee and the hedging strategy. We point out that quadratic pricing and hedging is very popular in financial mathematics and we would like to mention recent papers by Ankirchner and Heine (2012), Fujii and Takahashi (2014), Jeanblanc et. al. (2012), Kharroubi et. al. (2013), Kohlmann et. al. (2010) where backward stochastic differential equations are used.

This paper is structured as follows. In Section 2 we describe the model. In Section 3 we solve a quadratic optimization problem and in Section 4 the solution of the quadratic

optimization problem is used to solve the pricing and hedging problem for variable annuities with state-dependent fees. In Section 5 we present a numerical example which illustrates how our solution can be applied in practice. In the numerical example the dependence between Lévy processes is modelled with a Lévy Clayton copula.

2 The model

We deal with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and a finite time horizon $T < \infty$. We assume that \mathcal{F} satisfies the usual hypotheses of completeness (\mathcal{F}_0 contains all sets of \mathbb{P} -measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define an \mathcal{F} -adapted, two-dimensional Lévy process $L = (L_F, L_S) = (L_F(t), L_S(t), 0 \leq t \leq T)$. Its discontinuous part is denoted by $L^d = (L_F^d, L_S^d)$.

The financial market consists of a risk-free bank account $R = (R(t), 0 \leq t \leq T)$ and two risky assets $F = (F(t), 0 \leq t \leq T)$ and $S = (S(t), 0 \leq t \leq T)$. The value of the risk-free bank account satisfies the dynamics

$$R(t) = R(0)e^{rt}, \quad 0 \leq t \leq T, \quad (2.1)$$

and the prices of the risky assets are modelled with dependent exponential Lévy processes, i.e. they satisfy the dynamics

$$F(t) = F(0)e^{L_F(t)}, \quad S(t) = S(0)e^{L_S(t)}, \quad 0 \leq t \leq T.$$

By the Lévy-Itô decomposition, see Theorem 2.4.1 in Applebaum (2004), we can consider the representations

$$\begin{aligned} F(t) &= F(0)e^{\mu_F^* t + \sigma_{F,1} W(t) + \sigma_{F,2} B(t) + \int_0^t \int_{\mathbb{R}^2} z_F \tilde{N}(ds, dz_F, dz_S)}, \quad 0 \leq t \leq T, \\ S(t) &= S(0)e^{\mu_S^* t + \sigma_{S,1} W(t) + \sigma_{S,2} B(t) + \int_0^t \int_{\mathbb{R}^2} z_S \tilde{N}(ds, dz_F, dz_S)}, \quad 0 \leq t \leq T, \end{aligned} \quad (2.2)$$

where $W = (W(t), 0 \leq t \leq T)$ and $B = (B(t), 0 \leq t \leq T)$ are independent Brownian motions, and N is a random measure on $\Omega \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R}^2)$ which is independent of (W, B) . The compensated random measure \tilde{N} is defined by

$$\tilde{N}(dt, dz_F, dz_S) = N(dt, dz_F, dz_S) - \nu(dz_F, dz_S)dt,$$

where ν is a σ -finite measure on $\mathcal{B}(\mathbb{R}^2)$ called a Lévy measure. We set $N([0, T], \{(0, 0)\}) = \nu(\{(0, 0)\}) = 0$ and we assume that

$$\mathbf{(A)} \quad \int_{\mathbb{R}^2} (e^{2z_F} + e^{2z_S}) \nu(dz_F, dz_S) < \infty.$$

The random measure N counts the number of jumps of a given size of the Lévy process $L = (L_F, L_S)$, see Chapter 2.3 in Applebaum (2004). We point out that we use dependent Lévy process (L_F, L_S) to model the asset prices (F, S) . The continuous parts of the Lévy processes are correlated with coefficient ρ . Hence, by the Cholesky decomposition we can choose

$$\begin{aligned} \sigma_{S,1} &= \sigma_S, & \sigma_{S,2} &= 0, \\ \sigma_{F,1} &= \sigma_F \rho, & \sigma_{F,2} &= \sigma_F \sqrt{1 - \rho^2}. \end{aligned} \tag{2.3}$$

The dependence between the discontinuous parts of the Lévy processes is modelled with an appropriate form of the two-dimensional Lévy measure ν , see Chapter 5 in Cont and Tankov (2004) and Section 5.

By the Itô's formula we get the dynamics

$$\begin{aligned} \frac{dF(t)}{F(t-)} &= \mu_F dt + \sigma_{F,1} dW(t) + \sigma_{F,2} dB(t) + \int_{\mathbb{R}^2} (e^{z_F} - 1) \tilde{N}(dt, dz_F, dz_S), & 0 \leq t \leq T, \\ \frac{dS(t)}{S(t-)} &= \mu_S dt + \sigma_S dW(t) + \int_{\mathbb{R}^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S), & 0 \leq t \leq T, \end{aligned} \tag{2.4}$$

where the drifts μ_F and μ_S are appropriately defined, see Proposition 5.1.1 in Applebaum (2004), and the volatilities $\sigma_{F,1}$ and $\sigma_{F,2}$ satisfy (2.3). In the sequel we use (2.4). We shall assume that

(B) $\mu_F \geq r$ and $\mu_S \geq r$,

which is a classical assumption in financial models. Since the processes F and S solve linear SDEs, we conclude that $\mathbb{E}[|S(t)|^2] < K$, $\mathbb{E}[|F(t)|^2] < K$, $0 \leq t \leq T$, see Corollary 6.2.4 in Applebaum (2004). By square integrability of stochastic integrals, see Theorem 4.2.3 in Applebaum (2004), and the Doob's inequality we can also prove

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} |F(t)|^2\right] &\leq K \mathbb{E}\left[1 + \int_0^T |F(u-)|^2 du + \sup_{0 \leq t \leq T} \left| \int_0^t F(u-) \sigma_{F,1} dW(u) \right|^2 \right. \\ &\quad \left. + \sup_{0 \leq t \leq T} \left| \int_0^t F(u-) \sigma_{F,2} dB(u) \right|^2 + \sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{R}^2} F(u-) (e^{z^F} - 1) \tilde{N}(du, dz_F, dz_S) \right|^2 \right] \\ &\leq K \sup_{0 \leq t \leq T} \mathbb{E}\left[1 + \int_0^T |F(u-)|^2 du + \left| \int_0^t F(u-) \sigma_{F,1} dW(u) \right|^2 \right. \\ &\quad \left. + \left| \int_0^t F(u-) \sigma_{F,1} dW(u) \right|^2 + \left| \int_0^t \int_{\mathbb{R}^2} F(u-) (e^{z^F} - 1) \tilde{N}(du, dz_F, dz_S) \right|^2 \right] \leq K, \end{aligned} \quad (2.5)$$

and we deduce that $\mathbb{E}[\sup_{0 \leq t \leq T} |F(t)|^2] < \infty$ and $\mathbb{E}[\sup_{0 \leq t \leq T} |S(t)|^2] < \infty$.

The insurer can invest in the risk-free bank account R and in the risky asset S . The risky asset F is not traded in the financial market and it is the underlying investment fund for the variable annuity contract. Under the variable annuity contract the terminal benefit is linked to the performance of the investment fund F and a guaranteed terminal benefit is defined in the contract. In order to finance the guaranteed maturity benefit the insurer deducts fees from the policyholder's variable annuity account over the lifetime of the contract. The dynamics of the policyholder's account $V = (V(t), 0 \leq t \leq T)$ is given with the stochastic differential equation

$$\begin{aligned} dV(t) &= V(t-) \frac{dF(t)}{F(t-)} - g(V(t-)) dt \\ &= V(t-) (\mu_F dt + \sigma_{F,1} dW(t) + \sigma_{F,2} dB(t) + \int_{\mathbb{R}^2} (e^{z^F} - 1) \tilde{N}(dt, dz_F, dz_S)) \\ &\quad - g(V(t-)) dt, \quad 0 \leq t \leq T, \\ V(0) &= v > 0 \end{aligned} \quad (2.6)$$

where v denotes a premium invested by the policyholder and g denotes a state-dependent

fee deducted by the insurer. If $g(v) = \alpha v$, then we consider the classical case with a proportional fee. In this paper we are interested in more sophisticated state-dependent fees. In particular, we would like to study the case $g(v) = \alpha v \mathbf{1}\{v < v^*\}$ under which the fee is deducted from the account at a fixed rate α only if the account value is below a pre-specified barrier v^* . Such a payment scheme has been recently studied in Bernard et. al. (2014) in a Black-Scholes model. In this paper we consider a general state-dependent fee process g and we assume that

$$(C) \quad g(v) \geq 0, \quad |g(v)| \leq K|v|, \quad v \in \mathbb{R}.$$

Advantages of introducing state-dependent fees for variable annuity contracts are discussed in Introduction. Moreover, we have to assume that

$$(D) \quad \text{the SDE (2.6) has a unique positive solution } V \text{ such that } \mathbb{E}[\sup_{0 \leq t \leq T} |V(t)|^2] < \infty.$$

If a solution V exists to the SDE (2.6), then the solution V is positive and square integrable. Since we do not want to assume that g is Lipschitz continuous, the existence of a unique solution to the SDE (2.6) is a delicate issue.

Lemma 2.1. *If L_F is a finite variation Lévy process and (A),(C) are satisfied, then there exists a unique positive solution V to the SDE (2.6). Moreover, $\mathbb{E}[\sup_{0 \leq t \leq T} |V(t)|^2] < \infty$.*

Proof. Since g satisfies the linear growth condition, by the Itô's formula we get

$$V(t) = v e^{\mu_F^* t - \int_0^t \frac{g(V(s-))}{V(s-)} \mathbf{1}\{V(s-) \neq 0\} ds + \sigma_{F,1} dW(t) + \sigma_{F,2} dB(t) + \int_{\mathbb{R}^2} z_F \tilde{N}(t, dz_F, dz_S)}, \quad 0 \leq t \leq T.$$

Hence, any solution V to (2.6) is positive. Let $\mathcal{V}(t) = \ln V(t)$. We obtain the dynamics

$$d\mathcal{V}(t) = \mu_F^* dt - \frac{g(e^{\mathcal{V}(t-)})}{e^{\mathcal{V}(t-)}} dt + \sigma_{F,1} dW(t) + \sigma_{F,2} dB(t) + \int_{\mathbb{R}^2} z_F \tilde{N}(dt, dz_F, dz_S). \quad (2.7)$$

Since the drift in (2.7) is bounded, by Theorem 5.9 and Remark 5.16 in Meyer-Brandis and Proske (2006) there exists a unique solution \mathcal{V} to the SDE (2.7). Consequently,

there exists a unique positive solution V to the SDE (2.6). The square integrability of V can be proved by standard techniques in SDEs, see (2.5) or Proposition 2.5.1 in Delong (2013). \square

If e.g. the Lévy process L_F is a compound Poisson process, then Lemma 2.1 is satisfied. The result for a general Lévy process has not been proved to the best of our knowledge. However, it is pointed out in Meyer-Brandis and Proske (2010) that the existence of a unique solution to a SDE driven by a Lévy process with a bounded drift can be proved by using the method from Meyer-Brandis and Proske (2010) which is used to prove the existence of a unique solution to a SDE driven by a Brownian motion with a bounded drift.

Under the variable annuity contract with a guaranteed maturity benefit the insurer faces a liability $H(V(T))$ contingent on the policyholder's account value. The simplest example of a guaranteed maturity benefit would be the return of the premium at the terminal time of the contract in the case when the terminal account value drops below its initial value. In that case the guarantee H is a put option $H(V(t)) = (v - V(t))^+$. We consider guarantees H which satisfy the assumption

$$\mathbf{(E)} \quad H(v) \geq 0, \quad |H(v)| \leq K(1 + |v|), \quad v \in \mathbb{R}.$$

The insurer collects the fee g from the policyholder's account (2.6) and manages a hedging portfolio $X = (X(t), 0 \leq t \leq T)$ in order to hedge the issued guarantee H . Let $\pi = (\pi(t), 0 \leq t \leq T)$ denote a hedging strategy. By π we denote the amount of wealth which is invested into the risky asset S . The dynamics of the hedging portfolio X is given with the stochastic differential equation

$$\begin{aligned} dX^{\pi,x}(t) &= \pi(t) \frac{dS(t)}{S(t-)} + (X^{\pi,x}(t-) - \pi(t))r dt + g(V(t-))dt \\ &= \pi(t) (\mu_S dt + \sigma_S dW(t) + \int_{\mathbb{R}^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S)) \\ &\quad + (X^{\pi,x}(t-) - \pi(t))r dt + g(V(t-))dt, \quad 0 \leq t \leq T, \\ X^{\pi,x}(0) &= x, \end{aligned} \tag{2.8}$$

where x denotes an initial capital which is invested in the hedging portfolio by the insurer at the inception of the contract. The hedging portfolio (2.8) is financed with the insurer's initial capital x and the fee process g . Let us introduce the set of admissible hedging strategies.

Definition 2.1. *A strategy $\pi := (\pi(t), 0 \leq t \leq T)$ is called admissible, written $\pi \in \mathcal{A}$, if it satisfies the conditions:*

1. $\pi : [0, T] \times \Omega \rightarrow \mathbb{R}$ is an \mathcal{F} -predictable process,
2. $\mathbb{E} \left[\int_0^T |\pi(t)|^2 dt \right] < \infty$,
3. there exists a unique solution $X^{\pi, x}$ to the SDE (2.8).

Let us remark that the hedging portfolio is a square integrable process.

Lemma 2.2. *For an admissible hedging strategy $\pi \in \mathcal{A}$ the solution $X^{\pi, x}$ to the SDE (2.8) satisfies $\mathbb{E}[\sup_{t \in [0, T]} |X^{\pi, x}(t)|^2] < \infty$.*

Proof. Since (2.8) holds, we have

$$\begin{aligned} X^{\pi, x}(t) &= xe^{rt} + \int_0^t \pi(u)e^{r(t-u)}(\mu_S - r)du + \int_0^t \pi(u)e^{r(t-u)}\sigma_S dW(u) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} \pi(u)e^{r(t-u)}(e^{z_S} - 1)\tilde{N}(du, dz_F, dz_S) + \int_0^t e^{r(t-u)}g(V(u))du, \quad 0 \leq t \leq T. \end{aligned}$$

The square integrability of $X^{\pi, x}$ can be deduced from the admissability of π , square integrability of the stochastic integrals, see Theorem 4.2.3 in Applebaum (2004), the Doob's inequality and assumptions (C)-(D), see (2.5). \square

The insurer has to price and hedge the guaranteed maturity benefit embedded in the variable annuity contract. We have to choose the fee g and the hedging strategy π for the terminal liability H . Since we consider an incomplete financial market, we have a range of different objectives which can be used for pricing and hedging. We can use a quadratic objective under an equivalent martingale measure, a quadratic

objective under the real-world measure, local risk-minimization, a utility-based (risk-based) objective such as indifference pricing and hedging under exponential utility. All those objectives lead in our model to tractable mathematical optimization problems. We decide to use a quadratic objective under the real-world measure since it is most often applied in practice and does not depend on subjective parameters. Our first step is to find an admissible hedging strategy π which minimizes the quadratic loss resulting from the mismatch between the hedging portfolio and the liability:

$$\psi(x) = \min_{\pi \in \mathcal{A}} \mathbb{E} \left[|X^{\pi,x}(T) - H(V(T))|^2 \right], \quad (2.9)$$

and find an initial capital x for the hedging portfolio which minimizes the optimal quadratic loss $\psi(x)$. In the second step, we use the solution of our quadratic optimization problem to define the fee and the hedging strategy for the maturity guarantee embedded in the variable annuity contract. In this paper we neglect mortality risk but we would like to point out that our pricing and hedging problem can still be solved if mortality risk and guaranteed death benefits are taken into account in the model.

3 The Solution to the Quadratic Optimization Problem

In order to solve our quadratic optimization problem (2.9) we follow the approach based on Backward Stochastic Differential Equations (BSDEs), see Lim (2005), Øksendal and Hu (2008), Chapter 10.2 in Delong (2013). We sketch the idea of that approach. We consider two equations:

$$\begin{aligned} dY(t) &= -f(t)dt, \quad 0 \leq t \leq T, \\ Y(T) &= 1, \end{aligned} \quad (3.1)$$

and

$$\begin{aligned}
d\mathcal{Y}(t) &= -f'(t)dt + \mathcal{Z}_1(t)dW(t) + \mathcal{Z}_2(t)dB(t) \\
&\quad + \int_{\mathbb{R}^2} \mathcal{U}(t, z_F, z_S) \tilde{N}(dt, dz_F, dz_S), \quad 0 \leq t \leq T, \\
\mathcal{Y}(T) &= H(V(T)).
\end{aligned} \tag{3.2}$$

The functions f and f' will be specified in the sequel. Equation (3.1) is an ordinary differential equation with a terminal condition and equation (3.2) is a stochastic differential equation with a random terminal condition (called a backward stochastic differential equation). We introduce the process

$$\hat{\mathcal{Y}}(t) = -2Y(t)\mathcal{Y}(t), \quad 0 \leq t \leq T,$$

which has the dynamics

$$\begin{aligned}
d\hat{\mathcal{Y}}(t) &= 2(Y(t)f'(t) + \mathcal{Y}(t-)f(t))dt \\
&\quad - 2Y(t)\mathcal{Z}_1(t)dW(t) - 2Y(t)\mathcal{Z}_2(t)dB(t) \\
&\quad - 2 \int_{\mathbb{R}^2} Y(t)\mathcal{U}(t, z_F, z_S) \tilde{N}(dt, dz_F, dz_S), \quad 0 \leq t \leq T, \\
\hat{\mathcal{Y}}(T) &= -2H(V(T)).
\end{aligned}$$

Let $\pi \in \mathcal{A}$ denote an admissible hedging strategy and let us consider the hedging portfolio $X^{\pi, x}$ under the strategy π . By the Itô's formula we can derive the dynamics

$$\begin{aligned}
d(Y(t)(X^{\pi, x}(t))^2) &= Y(t) \left(2X^{\pi, x}(t-) \pi(t) (\mu_S dt + \sigma_S dW(t)) \right. \\
&\quad \left. + \int_{\mathbb{R}^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S) \right) \\
&\quad + 2X^{\pi, x}(t-) (X^{\pi, x}(t-) - \pi(t)) r dt + 2X^{\pi, x}(t-) g(V(t-)) dt \\
&\quad + |\pi(t)\sigma_S|^2 dt + \int_{\mathbb{R}^2} |\pi(t)|^2 (e^{z_S} - 1)^2 N(dt, dz_F, dz_S) - |X^{\pi, x}(t-)|^2 f(t) dt.
\end{aligned}$$

and

$$\begin{aligned}
d(\hat{\mathcal{Y}}(t)X^{\pi,x}(t)) &= \hat{\mathcal{Y}}(t-)\left(\pi(t)(\mu_S dt + \sigma_S dW(t) + \int_{\mathbb{R}^2} (e^{z_S} - 1)\tilde{N}(dt, dz_F, dz_S))\right. \\
&\quad \left.+ (X^{\pi,x}(t-) - \pi(t))r dt + g(V(t-))dt\right) \\
&\quad + X^{\pi,x}(t-)\left(2(Y(t)f'(t) + \mathcal{Y}(t-)f(t))dt - 2Y(t)\mathcal{Z}_1(t)dW(t)\right. \\
&\quad \left.- 2Y(t)\mathcal{Z}_2(t)dB(t) - 2\int_{\mathbb{R}^2} Y(t)\mathcal{U}(t, z_F, z_S)\tilde{N}(dt, dz_F, dz_S)\right) \\
&\quad - 2Y(t)\mathcal{Z}_1(t)\pi(t)\sigma_S dt - 2\int_{\mathbb{R}^2} Y(t)\mathcal{U}(t, z_F, z_S)(e^{z_S} - 1)N(dt, dz_F, dz_S).
\end{aligned}$$

Taking the expectation and combining the terms, we can deduce the formula for the quadratic loss:

$$\begin{aligned}
\mathbb{E}[|X^{\pi,x}(T) - H(V(T))|^2] &= \mathbb{E}\left[|Y(T)|X^{\pi,x}(T) - \mathcal{Y}(T)|^2\right] \\
&= \mathbb{E}\left[|Y(T)|X^{\pi,x}(T)|^2 + \hat{\mathcal{Y}}(T)X^{\pi,x}(T) + Y(T)|\mathcal{Y}(T)|^2\right] \\
&= Y(0)x^2 + \hat{\mathcal{Y}}(0)x \\
&\quad + \mathbb{E}\left[\int_0^T Y(t)\sigma^2\left\{\pi(t) + \frac{\mu_S - r}{\sigma^2}X^{\pi,x}(t-)\right.\right. \\
&\quad \left.\left.- \frac{\mathcal{Z}_1(t)\sigma_S + \int_{\mathbb{R}^2}\mathcal{U}(t, z_F, z_S)(e^{z_S} - 1)\nu(dz_F, dz_S)}{\sigma^2} + \frac{\hat{\mathcal{Y}}(t-)}{2Y(t)}\frac{\mu_S - r}{\sigma^2}\right\}^2 dt\right. \\
&\quad + \int_0^T |X^{\pi,x}(t-)|^2\left\{-f(t) + 2Y(t)r - \left|\frac{\mu_S - r}{\sigma}\right|^2 Y(t)\right\}dt \\
&\quad + \int_0^T X^{\pi,x}(t-)\left\{2Y(t)f'(t) + 2\mathcal{Y}(t-)f(t) + \hat{\mathcal{Y}}(t-)r + 2Y(t)g(V(t-))\right. \\
&\quad \left.- 2Y(t)(\mu_S - r)\left(-\frac{\mathcal{Z}_1(t)\sigma_S + \int_{\mathbb{R}^2}\mathcal{U}(t, z_F, z_S)(e^{z_S} - 1)\nu(dz_F, dz_S)}{\sigma^2} + \frac{\hat{\mathcal{Y}}(t-)}{2Y(t)}\frac{\mu_S - r}{\sigma^2}\right)\right\}dt\right] \\
&\quad + \mathbb{E}\left[|Y(T)|\mathcal{Y}(T)|^2 + \int_0^T \hat{\mathcal{Y}}(t-)g(V(t-))dt\right. \\
&\quad \left.- \int_0^T Y(t)\sigma^2\left\{-\frac{\mathcal{Z}_1(t)\sigma_S + \int_{\mathbb{R}^2}\mathcal{U}(t, z_F, z_S)(e^{z_S} - 1)\nu(dz_F, dz_S)}{\sigma^2} + \frac{\hat{\mathcal{Y}}(t-)}{2Y(t)}\frac{\mu_S - r}{\sigma^2}\right\}^2 dt\right],
\end{aligned}$$

where

$$\sigma^2 = |\sigma_S|^2 + \int_{\mathbb{R}^2} (e^{z_S} - 1)^2 \nu(dz_F, dz_S).$$

We choose the strategy π and the functions f, \hat{f} which make the first three expectation vanish. We now state the key result of this section.

Theorem 3.1. *Let (A)-(E) hold. We consider the equations:*

$$\begin{aligned} dY(t) &= Y(t) \left(-2r + \frac{(\mu_S - r)^2}{\sigma^2} \right) dt, \quad 0 \leq t \leq T, \\ Y(T) &= 1, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d\mathcal{Y}(t) &= \left(\mathcal{Y}(t-)r + g(V(t-)) \right. \\ &\quad \left. + \frac{\mu_S - r}{\sigma^2} (\mathcal{Z}_1(t)\sigma_S + \int_{\mathbb{R}^2} \mathcal{U}(t, z_F, z_S)(e^{z_S} - 1)\nu(dz_F, dz_S)) \right) dt \\ &\quad + \mathcal{Z}_1(t)dW(t) + \mathcal{Z}_2(t)dB(t) + \int_{\mathbb{R}^2} \mathcal{U}(t, z_F, z_S)\tilde{N}(dt, dz_F, dz_S), \quad 0 \leq t \leq T, \\ \mathcal{Y}(T) &= H(V(T)). \end{aligned} \tag{3.4}$$

(i) *There exists unique solutions Y and $(\mathcal{Y}, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{U})$ to equations (3.3)-(3.4). Moreover, we have*

$$\begin{aligned} \mathbb{E}[\sup_{0 \leq t \leq T} |\mathcal{Y}(t)|^2] &< \infty, \quad \mathbb{E}\left[\int_0^T |\mathcal{Z}_1(t)|^2 dt\right] < \infty, \\ \mathbb{E}\left[\int_0^T |\mathcal{Z}_2(t)|^2 dt\right] &< \infty, \quad \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^2} |\mathcal{U}(t, z_F, z_S)|^2 \nu(dz_F, dz_S) dt\right] < \infty. \end{aligned}$$

(ii) *The optimal admissible hedging strategy π^* for the quadratic loss (2.9) is given by*

$$\begin{aligned} \pi^*(t) &= \frac{\mathcal{Z}_1(t)\sigma_S + \int_{\mathbb{R}^2} \mathcal{U}(t, z_F, z_S)(e^{z_S} - 1)\nu(dz_F, dz_S)}{\sigma^2} \\ &\quad + \frac{\mu_S - r}{\sigma^2} (\mathcal{Y}(t-) - X^{\pi^*, x}(t-)), \quad 0 \leq t \leq T, \end{aligned} \tag{3.5}$$

where the optimal hedging portfolio $X^{\pi^*,x}$ satisfies the dynamics

$$\begin{aligned} dX^{\pi^*,x}(t) &= \pi^*(t)(\mu_S dt + \sigma_S dW(t) + \int_{\mathbb{R}^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S)) \\ &\quad + (X^{\pi^*,x}(t-) - \pi^*(t)) r dt + g(V(t-)) dt, \quad 0 \leq t \leq T, \\ X^{\pi^*,x}(0) &= x. \end{aligned} \tag{3.6}$$

(iii) The optimal initial capital for the hedging portfolio $X^{\pi^*,x}$ which minimizes the optimal quadratic loss (2.9) is given by

$$x^* = \mathcal{Y}(0). \tag{3.7}$$

Proof. The result can be proved by following closely the proofs from Section 3 in Lim (2005), the proof of Theorem 2.1 in Øksendal and Hu (2008) or the proofs from Chapter 10.2 in Delong (2013). Details can be obtained from the author upon the request. In addition, we can conclude that the control process \mathcal{U} is independent of z_S , i.e. we have $\mathcal{U}(t, z_F, z_S) = \mathcal{U}(t, z_F)$. \square

In order to apply the optimal hedging strategy (3.5) and calculate the optimal initial capital (3.7) we need to solve the backward stochastic differential equation (3.4). In our general case the BSDE (3.4) has to be solved numerically. We comment how the solution $(\mathcal{Y}, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{U})$ can be derived numerically, see Chapter 5.1 in Delong (2013) for details. First, we introduce a partition $0 = t_0 < t_1 < \dots < t_i < \dots < t_n = T$ of the time interval $[0, T]$ with a time step h . Next, the solution can be defined by the recursive

relation

$$\begin{aligned}
\mathcal{Y}(T) &= H(V(T)), \\
\mathcal{Z}_1(t_i) &= \frac{1}{h} \mathbb{E} \left[\mathcal{Y}(t_{i+1}) (W(t_{i+1}) - W(t_i)) | \mathcal{F}_{t_i} \right], \quad i = 0, \dots, n-1, \\
\mathcal{Z}_2(t_i) &= \frac{1}{h} \mathbb{E} \left[\mathcal{Y}(t_{i+1}) (B(t_{i+1}) - B(t_i)) | \mathcal{F}_{t_i} \right], \quad i = 0, \dots, n-1, \\
&\int_{\mathbb{R}^2} \mathcal{U}(t_i, z_F, z_S) (e^{z_S} - 1) \nu(dz_F, dz_S) \\
&= \frac{1}{h} \mathbb{E} \left[\mathcal{Y}(t_{i+1}) \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S) | \mathcal{F}_{t_i} \right], \quad i = 0, \dots, n-1, \\
\mathcal{Y}(t_i) &= \frac{1}{1+rh} \mathbb{E} \left[\mathcal{Y}(t_{i+1}) - \left(g(V(t_i)) + \frac{\mu_S - r}{\sigma^2} \mathcal{Z}_1(t_i) \sigma_S \right. \right. \\
&\quad \left. \left. + \frac{\mu_S - r}{\sigma^2} \int_{\mathbb{R}^2} \mathcal{U}(t_i, z_F, z_S) (e^{z_S} - 1) \nu(dz_F, dz_S) \right) h | \mathcal{F}_{t_i} \right], \quad i = 0, \dots, n-1, \quad (3.8)
\end{aligned}$$

see Bouchard and Elie (2008). Finally, the expectations in (3.8) are estimated by the Least Squares Monte Carlo method, i.e. are estimated by fitting regression polynomials at each point $(t_i)_{i=0, \dots, n-1}$ with a dependent variable $V(t_i)$ based on a generated sample of $(F(t_i))_{i=1, \dots, n}$, see Longstaff and Schwartz (2001). Notice that we do not need to estimate the solution \mathcal{Z}_2 to define the optimal strategy (3.5) and the optimal capital (3.7).

4 The Solution to the Pricing and Hedging Problem

In the previous section we have derived the optimal initial capital for the hedging portfolio and the optimal hedging strategy under the quadratic objective (2.9), see Theorem 3.1. We have answered the question how to hedge the pay-off from the guarantee embedded in the variable annuity contract. However, the question how to set the fee for the guarantee still remains open. Since the insurer does not want to incur any costs at the inception of the contract and only wants to use the collected fees g to cover the

terminal guarantee H , we should require that $x^* = \mathcal{Y}(0) = 0$. The condition

$$\mathcal{Y}(0) = 0, \quad (4.1)$$

can be called the pricing principle from which the fee g for the guaranteed benefit (or the price of the guaranteed benefit) can be calculated for the insurer who adopts the quadratic pricing and hedging objective (2.9).

In order to study the pricing equation (4.1) in detail, we need to investigate the process \mathcal{Y} . The next result points out important properties of the process \mathcal{Y} .

Theorem 4.1. *Let (A)-(E) hold and let the Lévy measure ν be absolutely continuous. We consider the BSDE (3.4) and the process $M := (M(t), 0 \leq t \leq T)$ given by*

$$\begin{aligned} \frac{dM(t)}{M(t-)} &= -\frac{\mu_S - r}{\sigma^2} \sigma_S dW(t) - \int_{\mathbb{R}^2} \frac{\mu_S - r}{\sigma^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S), \quad 0 \leq t \leq T, \\ M(0) &= 1. \end{aligned}$$

(i) *The process \mathcal{Y} has the representation*

$$\begin{aligned} \mathcal{Y}(t) &= \mathbb{E} \left[\frac{M(T)}{M(t)} e^{-r(T-t)} H(V(T)) \right. \\ &\quad \left. - \frac{M(T)}{M(t)} \int_t^T e^{-r(s-t)} g(V(s)) ds \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \end{aligned} \quad (4.2)$$

(ii) *The process $(M(t)S(t)e^{-rt})_{0 \leq t \leq T}$ is a martingale.*

Proof. From the theory of SDEs we deduce that the process M is a square integrable martingale, see Proposition 8.23 in Cont and Tankov (2004) and (2.5). Let us introduce a Lévy process \mathcal{L} defined by

$$\mathcal{L}(t) = -\frac{\mu_S - r}{\sigma^2} \sigma_S W(t) - \int_0^t \int_{\mathbb{R}^2} \frac{\mu_S - r}{\sigma^2} (e^{z_S} - 1) \tilde{N}(ds, dz_F, dz_S), \quad 0 \leq t \leq T. \quad (4.3)$$

By Proposition 5.1.1 in Applebaum (2004) we have

$$M(t) = e^{\mathcal{L}(t) - \frac{1}{2}(\frac{\mu_S - r}{\sigma^2} \sigma_S)^2 t} \prod_{0 \leq s \leq t} (1 + \Delta \mathcal{L}(s)) e^{-\Delta \mathcal{L}(s)}, \quad 0 \leq t \leq T,$$

where $\Delta \mathcal{L}(s) = \mathcal{L}(s) - \mathcal{L}(s-)$. Since the Lévy measure ν is absolutely continuous, we have $\nu(\{z_S : \frac{\mu_S - r}{\sigma^2}(e^{z_S} - 1) = 1\}) = 0$ and $\mathbb{P}(\Delta \mathcal{L}(t) = -1 | \Delta \mathcal{L}(t) \neq 0) = 0$ for any $t \in [0, T]$. Consequently, $\mathbb{P}(\inf_{0 \leq t \leq T} |M(t)| > 0) = 1$. Let us now consider the process

$$\mathcal{Y}^*(t) = e^{-rt} \mathcal{Y}(t) - \int_0^t e^{-rs} g(V(s)) ds, \quad 0 \leq t \leq T.$$

The process \mathcal{Y}^* is square integrable since \mathcal{Y} is square integrable, see Theorem 3.1, and (C)-(D) hold. By the Itô's formula we get the dynamics

$$\begin{aligned} d(\mathcal{Y}^*(t)M(t)) &= \mathcal{Y}^*(t-)M(t-) \left(-\frac{\mu_S - r}{\sigma^2} \sigma_S dW(t) \right. \\ &\quad \left. - \int_{\mathbb{R}^2} \frac{\mu_S - r}{\sigma^2} (e^{z_S} - 1) \tilde{N}(dt, dz_F, dz_S) \right) \\ &\quad + M(t-) \left(e^{-rt} \mathcal{Z}_1(t) dW(t) + e^{-rt} \mathcal{Z}_2(t) dB(t) \right. \\ &\quad \left. + \int_{\mathbb{R}^2} \left(1 - \frac{\mu_S - r}{\sigma^2} (e^{z_S} - 1) \right) e^{-rt} \mathcal{U}(t, z_F, z_S) \tilde{N}(dt, dz_F, dz_S) \right), \end{aligned}$$

and we can deduce that $\mathcal{Y}^*(t)M(t)$ is a local martingale since it is driven by stochastic integrals with respect to Brownian motions and a compensated Poisson random measure, see Theorem 4.2.12 in Applebaum (2004). Moreover, we know that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\mathcal{Y}^*(t)M(t)| \right] \leq \frac{1}{2} \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |\mathcal{Y}^*(t)|^2 \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |M(t)|^2 \right] \right) < \infty,$$

and we can conclude that $\mathcal{Y}^*(t)M(t)$ is a true martingale. Hence, we have the representation

$$\mathcal{Y}^*(t)M(t) = \mathbb{E}[\mathcal{Y}^*(T)M(T) | \mathcal{F}_t],$$

and assertion (i) is proved. By the Itô's formula we can also derive the dynamics

$$\begin{aligned} d(M(t)S(t)e^{-rt}) &= M(t-)S(t-)e^{-rt} \left(\left(1 - \frac{\mu_S - r}{\sigma^2}\right) \sigma_S dW(t) \right. \\ &\quad \left. + \int_{\mathbb{R}^2} \left(\left(1 - \frac{\mu_S - r}{\sigma^2}\right) (e^{zs} - 1) - \frac{\mu_S - r}{\sigma^2} (e^{zs} - 1)^2 \right) \tilde{N}(dt, dz_F, dz_S) \right), \end{aligned}$$

and, as for assertion (i), we can conclude that $M(t)S(t)e^{-rt}$ is a martingale and assertion (ii) is proved. \square

We point out that we can use the representation (4.2) in the Least Squares Monte Carlo to derive the solution \mathcal{Y} to the BSDE (3.4).

From the pricing principle (4.1) and Theorem 4.1 we deduce that the fee g should be set by the insurer at the inception of the contract in accordance with the condition

$$\mathbb{E} \left[M(T) e^{-rT} H(V(T)) \right] = \mathbb{E} \left[M(T) \int_0^T e^{-rs} g(V(s)) ds \right], \quad (4.4)$$

which tells us that the fee g should be set in accordance with the principle which guarantees the equivalence between the collected fees and the pay-off from the guarantee embedded in the variable annuity contract. Notice that the fee g affects both sides of equation (4.4) and, consequently, the fee g is a solution to a fixed point equation. The condition which defines the fee g is intuitively clear. Indeed, the fees should finance the guarantee. In formula (4.4) we see that the process M plays the role of the deflator for the cash flows. It is tempting to change the measure in (4.4) by introducing the measure

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = M(t), \quad 0 \leq t \leq T. \quad (4.5)$$

However, the process M is not a strictly positive martingale and the measure $\tilde{\mathbb{Q}}$ defined in (4.5) is not an equivalent probability measure for \mathbb{P} (and the process M is not a proper deflator). Hence, an equivalent martingale measure which should be used for arbitrage-free pricing cannot be defined in our model. The measure $\tilde{\mathbb{Q}}$ defined in (4.5)

is called a signed martingale measure for the traded asset S (since $\tilde{\mathbb{Q}}$ is absolutely continuous with respect to \mathbb{P} , $(M(t), 0 \leq t \leq T)$ is a square integrable martingale and $(M(t)S(t)e^{-rt}, 0 \leq t \leq T)$ is a martingale). If a signed martingale measure M is given, then a claim ξ is priced with the formula $\mathbb{E}[M(T)e^{-rT}\xi]$, see Schweizer (1996). In our general model with jumps the pricing principle (4.4) may lead to arbitrage opportunities. In order to avoid arbitrage, the insurer is likely to apply a pricing principle different from (4.4) to set the fee g . If a different pricing principle is applied to set the fee g , then the insurer resigns from adopting the quadratic pricing objective but he can still adopt the quadratic hedging objective and use the strategy (3.5) to hedge optimally in a mean-square sense the guaranteed maturity benefit. Regardless of the pricing principle, the optimality of the hedging strategy (3.5) holds under the quadratic objective (2.9).

We now comment when the pricing principle (4.4) is arbitrage-free. In some special cases the martingale M is strictly positive and the real-world measure \mathbb{P} can be changed to an equivalent martingale measure \mathbb{Q}^* .

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold and let the Lévy process L_S have only jumps smaller than $\ln\left(1 + \frac{\sigma^2}{\mu_S - r}\right)$, i.e. $\nu(\{z_S : \frac{\mu_S - r}{\sigma^2}(e^{z_S} - 1) > 1\}) = 0$.*

(i) *The process \mathcal{Y} has the representation*

$$\mathcal{Y}(t) = \mathbb{E}^{\mathbb{Q}^*} \left[e^{-r(T-t)} H(V(T)) - \int_t^T e^{-r(s-t)} g(V(s)) ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T,$$

under an equivalent probability measure \mathbb{Q}^ defined by the Radon-Nikodym derivative*

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = M(t), \quad 0 \leq t \leq T.$$

(ii) *The process $(S(t)e^{-rt}, 0 \leq t \leq T)$ is a \mathbb{Q}^* -martingale.*

Proof. Recalling (4.3), we can conclude that $\mathbb{P}(\Delta\mathcal{L}(t) > -1 \mid \Delta\mathcal{L}(t) \neq 0) = 1$ for any

$t \in [0, T]$ and, consequently, the martingale M is strictly positive. The results now follow from Theorem 4.1 and the Bayes formula. \square

The measure \mathbb{Q}^* defined in Theorem 4.2 is an equivalent martingale measure for our incomplete financial market (2.2).

In many applications a Lévy process with negative jumps (a spectrally negative Lévy process) is used to model crashes in the financial market. If a Lévy process L_S with negative jumps is used, then the assumptions of Theorem 4.3 are satisfied. Under the assumptions of Theorem 4.2 the fee g should be set by the insurer at the inception of the contract in accordance with the condition

$$\mathbb{E}^{\mathbb{Q}^*} \left[e^{-rT} H(V(T)) \right] = \mathbb{E}^{\mathbb{Q}^*} \left[\int_0^T e^{-rs} g(V(s)) ds \right], \quad (4.6)$$

which is a classical arbitrage-free pricing formula. If the pricing principle (4.6) and the optimal hedging strategy (3.5) are applied, then the insurer adopts the quadratic pricing and hedging objective (2.9).

It is worth pointing out that under the equivalent martingale measure \mathbb{Q}^* the processes

$$\begin{aligned} dW^{\mathbb{Q}^*}(t) &= dW(t) + \frac{\mu_S - r}{\sigma^2} \sigma_S dt, \\ dB^{\mathbb{Q}^*}(t) &= dB(t), \\ \tilde{N}^{\mathbb{Q}^*}(dt, dz_F, dz_S) &= N(dt, dz_F, dz_S) - \left(1 - \frac{\mu_S - r}{\sigma^2} (e^{z_S} - 1)\right) \nu(dz_F, dz_S) dt, \end{aligned}$$

are Brownian motions and a compensated Poisson random measure, see Theorem 1.32 and Lemma 1.33 in Øksendal and Sulem (2004). After the change of measure we deal

with the dynamics

$$\begin{aligned}
\frac{dS(t)}{S(t-)} &= rdt + \sigma_S dW^{\mathbb{Q}^*}(t) \\
&\quad + \int_{(-\infty, \infty) \times (-\infty, \ln(1 + \frac{\sigma^2}{\mu_S - r})} (e^{z_S} - 1) \tilde{N}^{\mathbb{Q}^*}(dt, dz_F, dz_S), \quad 0 \leq t \leq T, \\
\frac{dF(t)}{F(t-)} &= \left(\mu_F - \frac{\mu_S - r}{\sigma^2} \sigma_S \sigma_{F,1} \right. \\
&\quad - \int_{(-\infty, \infty) \times (-\infty, \ln(1 + \frac{\sigma^2}{\mu_S - r})} \frac{\mu_S - r}{\sigma^2} (e^{z_S} - 1) (e^{z_F} - 1) \nu(dz_F, dz_S) \Big) dt \\
&\quad + \sigma_{F,1} dW^{\mathbb{Q}^*}(t) + \sigma_{F,2} dB^{\mathbb{Q}^*}(t) \\
&\quad + \int_{(-\infty, \infty) \times (-\infty, \ln(1 + \frac{\sigma^2}{\mu_S - r})} (e^{z_F} - 1) \tilde{N}^{\mathbb{Q}^*}(dt, dz_F, dz_S), \quad 0 \leq t \leq T.
\end{aligned}$$

We now would like to interpret the optimal hedging strategy (3.5). Based on Theorems 4.1-4.2 and the discussion following that theorems, we can interpret the process \mathcal{Y} as the expected loss of the insurer which arises from the guarantee H , which will be paid in the future, and the fees g , which will be collected in the future. Hence, the process \mathcal{Y} can be interpreted as the price of the net liability. The process \mathcal{Y} can be interpreted as an arbitrage or an arbitrage-free price process depending on whether the assumptions of Theorem 4.2 are satisfied. Recalling results on BSDEs, see Corollary 4.1 in El Karoui et al. (1997) and Proposition 4.2 in Bouchard and Elie (2008) or Theorem 4.1.4 in Delong (2013), we can deduce that the process \mathcal{Z}_1 defines the change in the price of the net liability resulting from continuous changes in the account value V (resulting from changes in the Brownian motion W) and the process \mathcal{U} defines the change in the price of the net liability resulting from discontinuous changes in the account value V (resulting from changes in the Lévy process L_F^d). Consequently, the first term in the optimal hedging strategy (3.5) is the delta-hedging strategy. Since the insurer manages its hedging portfolio X^π to cover the pay-off from the guarantee H and should construct the hedging portfolio X^π which follows the price process of the net liability \mathcal{Y} , the second term in the optimal hedging strategy (3.5) is a correction factor adjusting the discrepancies between the optimal hedging portfolio X^{π^*} and the price of the net

liability \mathcal{Y} .

5 Numerical Example

In this last section we present a numerical example which illustrates how our results can be applied in practice. We consider a variable annuity contract under which the insurer guarantees to protect the premium v invested by the policyholder into the fund F . The insurer faces the guarantee of the form $H(V(T)) = (v - V(T))^+$ where V denotes the policyholder's account value. The time horizon is $T = 1$. We investigate two forms of state-dependent fees. The insurer deducts the fee g from the policyholder's account at a fixed rate α but only if the account value V is below a barrier v^* , i.e. the insurer uses a state-dependent fee of the form $g(v) = \alpha v \mathbf{1}\{v < v^*\}$. Alternatively, the insurer deducts the fee g from the policyholder's account at a fixed base rate α if the account value V is below a barrier v^* and at a fixed reduced rate $\beta\alpha$ if the account value V is above a barrier v^* , i.e. the insurer uses a state-dependent fee of the form $g(v) = \alpha v \mathbf{1}\{v < v^*\} + \beta\alpha v \mathbf{1}\{v \geq v^*\}$, $\beta \in (0, 1)$.

In our financial model (2.2) we have to specify the jumps component of the Lévy process (L_F, L_S) . We assume that that the discontinuous parts of the Lévy processes (L_F^d, L_S^d) are dependent compound Poisson processes which can only have negative jumps. We describe how we model the absolute values of the jumps of (L_F^d, L_S^d) . The margin L_F^d is a compound Poisson process with intensity λ_F and exponentially distributed jumps with expectation $1/\theta_F$, and the margin L_S^d is a compound Poisson process with intensity λ_S and exponentially distributed jumps with expectation $1/\theta_S$. The dependence between L_F^d and L_S^d is modelled with the Clayton Lévy copula

$$C(f, s) = \left(f^{-\theta} + s^{-\theta} \right)^{-1/\theta}, \quad (f, s) \in [0, \infty) \times [0, \infty),$$

which defines the tail integral of the Lévy measure for the two-dimensional Lévy process (L_F^d, L_S^d) in terms of the tail integrals of the Lévy measures of the one-dimensional

Lévy processes L_F^d and L_S^d , see Chapter 5.5 in Cont and Tankov (2004). If $\theta \rightarrow \infty$ we obtain perfect positive dependence of the margins, if $\theta \rightarrow 0$ we obtain independence of the margins. The Lévy measure ν of the two-dimensional Lévy process (L_F^d, L_S^d) is characterized with the density

$$\begin{aligned} \nu(dz_F, dz_S) &= C_{f_s}(\lambda_F e^{-\theta_F z_F}, \lambda_S e^{-\theta_S z_S}) \\ &\quad \cdot \lambda_F \lambda_S \theta_F e^{-\theta_F z_F} \theta_S e^{-\theta_S z_S} dz_F dz_S, \quad (z_F, z_S) \in [0, \infty)^2 - \{(0, 0)\}, \end{aligned}$$

where C_{f_s} denotes a partial derivative. In order to simulate the jumps of (L_F^d, L_S^d) , we use the following decomposition

$$\begin{aligned} L_F^d(t) &= L_F^\perp(t) + L_F^\parallel(t), \quad 0 \leq t \leq T, \\ L_S^d(t) &= L_S^\perp(t) + L_S^\parallel(t), \quad 0 \leq t \leq T, \end{aligned}$$

where L_F^\perp, L_S^\perp are independent compound Poisson processes with the intensities $\lambda_F - C(\lambda_F, \lambda_S), \lambda_S - C(\lambda_F, \lambda_S)$ and the jump tail distributions

$$\begin{aligned} Pr(Z_F^\perp > z) &= \frac{\lambda_F e^{-\theta_F z} - C(\lambda_F e^{-\theta_F z}, \lambda_S)}{\lambda_F - C(\lambda_F, \lambda_S)}, \quad z > 0, \\ Pr(Z_S^\perp > z) &= \frac{\lambda_S e^{-\theta_S z} - C(\lambda_F, \lambda_S e^{-\theta_S z})}{\lambda_S - C(\lambda_F, \lambda_S)}, \quad z > 0, \end{aligned}$$

and $(L_F^\parallel, L_S^\parallel)$ is an independent compound Poisson process with margins that jump at the same time with the intensity $C(\lambda_F, \lambda_S)$ and the jump tail distributions

$$\begin{aligned} Pr(Z_F^\parallel > z) &= \frac{C(\lambda_F e^{-\theta_F z}, \lambda_S)}{C(\lambda_F, \lambda_S)}, \quad z > 0, \\ Pr(Z_S^\parallel > z | Z_F^\parallel = y) &= \frac{C_f(\lambda_F e^{-\theta_F y}, \lambda_S e^{-\theta_S z})}{C_f(\lambda_F e^{-\theta_F y}, \lambda_S)}, \quad z > 0, \end{aligned}$$

where C_f denotes a partial derivative. For details we refer to Chapter 5.5 in Cont and Tankov (2004).

In Table 1 the values of the parameters are given. The guaranteed benefit is 100.

Table 1: The values of the parameters

The parameter	The value	The parameter	The value
v	100	r	0.03
μ_F	0.09	λ_F	0.4
μ_S	0.07	λ_S	0.6
σ_F	0.15	θ_F	10
σ_S	0.1	θ_S	12

We price and hedge our guarantee $H = (100 - V(1))^+$ by applying the results from the previous sections. Since the Lévy processes (L_F, L_S) have only negative jumps, the pricing principle (4.4) defines an arbitrage-free fee g . First, we assume that the insurer deducts the fee $g(v) = \alpha v \mathbf{1}\{v < v^*\}$. We study how the optimal fee rate α^* depends on the barrier v^* , above which the fee is not deducted from the account, and how the optimal fee rate α^* depends on the parameters (ρ, θ) , which model the dependence of the Lévy processes. The case with $\rho = 0$, $\theta = 0.01$ can be interpreted as a weak dependence of the Lévy processes L_F and L_S , the case with $\rho = 0.5$, $\theta = 1$ can be interpreted as a moderate dependence, and the case with $\rho = 0.9$, $\theta = 5$ can be interpreted as a strong dependence. The optimal fee rate α^* is derived from the pricing condition (4.4) by estimating the expectations by Monte Carlo method and solving the fixed point equation iteratively. The results based on 10000 samples are presented in Table 2. It is clear that the lower the barrier v^* is, the higher the optimal fee rate α^* is. The case with $v^* = 200$ can be interpreted as the classical case in which a proportional fee is deducted from the account, i.e. $g(v) = \alpha v$, since under the parameters from Table 1 there is a negligible probability that the account value exceeds 200. We can notice that the optimal fee rates α^* for the barrier $v^* = 100$ are very high compared to the optimal fee rates α^* for $v^* = 200$ (the optimal fee rates for the case without a barrier) and the insurer is not likely to issue the variable annuity contract under which the fee is deducted only if the account value is below the guaranteed benefit. However, it might be a good strategic decision to issue the contract under which the fee is deducted only if the account value is below the barrier $v^* = 110$, which is only slightly higher than

the guaranteed benefit. We can observe that the optimal fee rates α^* which correspond to the barrier $v^* = 110$ are close to the optimal fee rates α^* which correspond to $v^* = 200$ (the optimal fee rates for the case without a barrier). Other calculations also clearly confirm that introducing a barrier which is close to the guaranteed benefit increases the fee rate to levels which should be acceptable by policyholders. However, introducing a barrier which is equal to the guaranteed benefit increases the fee rate to levels which would not be acceptable by policyholders. In Table 2 we can also notice that the stronger the dependence between the fund F and the stock S is, the higher the optimal fee rate α^* is. This conclusion agrees with the applied quadratic objective and the intuition. If we want to hedge a liability F with an instrument S , than the higher the correlation between F and S is, the more units of S we would like to keep to hedge F . Consequently, a higher fee has to be collected in order to buy more units of S .

Table 2: The optimal fee rate α^* as a function of (ρ, θ, v^*) . The case of $g(v) = \alpha v \mathbf{1}\{v < v^*\}$.

The values of the parameters	The optimal fee rate
$\rho = 0, \theta = 0.01, v^* = 100$	0.216
$\rho = 0, \theta = 0.01, v^* = 110$	0.072
$\rho = 0, \theta = 0.01, v^* = 120$	0.053
$\rho = 0, \theta = 0.01, v^* = 200$	0.045
$\rho = 0.5, \theta = 1, v^* = 100$	0.346
$\rho = 0.5, \theta = 1, v^* = 110$	0.114
$\rho = 0.5, \theta = 1, v^* = 120$	0.087
$\rho = 0.5, \theta = 1, v^* = 200$	0.077
$\rho = 0.9, \theta = 5, v^* = 100$	0.785
$\rho = 0.9, \theta = 5, v^* = 110$	0.189
$\rho = 0.9, \theta = 5, v^* = 120$	0.148
$\rho = 0.9, \theta = 5, v^* = 200$	0.135

We also study the performance of the optimal hedging portfolio. The performance is estimated based on 10000 samples. In Table 3 we present the expected loss and the 95%-Value-at-Risk of the loss of the insurer's hedging portfolio under the optimal fee rate α^* from Table 2 and the optimal hedging strategy (3.5) after paying the claim

$H(V(1)) = (100 - V(1))^+$ from the guarantee. In order to apply our hedging strategy, we have to solve the BSDE (3.4). We use the Least Squares Monte Carlo (3.8) to estimate the processes $\mathcal{Y}, \mathcal{Z}_1, \mathcal{U}$. We use natural cubic splines for regression. Since the negative sign of the loss is interpreted as a gain, we conclude that under the optimal fee rate and the optimal hedging strategy the contract is expected to be profitable. In Table 3 we observe an interesting property. Except the case of the Value-at-Risk for $\rho = 0.9, \theta = 5$, we can see that the expected loss and the Value-at-Risk of the loss of the insurer's hedging portfolio under the optimal fee rate and the optimal hedging strategy are smaller for the barrier $v^* = 100$ than for the barrier $v^* = 200$ (which is interpreted as the classical case without a barrier). This property has been also observed in other calculations. The case of the Value-at-Risk for $\rho = 0.9, \theta = 5$ seems to be different from the other cases since in this case a huge fee rate ($\alpha^* = 0.785$) must be applied by the insurer who deducts $g(v) = \alpha v \mathbf{1}\{v < 100\}$. We can conclude that introducing a barrier above which the fee is not deducted from the account (by still keeping the fee rate at reasonable levels) is beneficial to the insurer from the point of view of hedging the guarantee. This hypothesis deserves more investigation and such an investigation is beyond the scope of this paper.

Table 3: The expected loss and the 95%-Value-at-Risk of the loss of the insurer's optimal hedging portfolio after paying the guarantee. The case of $g(v) = \alpha^* v \mathbf{1}\{v < v^*\}$.

The values of the parameters	The expected loss	The 95%-Value-at-Risk of the loss
$\rho = 0, \theta = 0.01, v^* = 100$	-0.728	15.439
$\rho = 0, \theta = 0.01, v^* = 200$	-0.431	16.442
$\rho = 0.5, \theta = 1, v^* = 100$	-2.052	15.762
$\rho = 0.5, \theta = 1, v^* = 200$	-1.482	16.163
$\rho = 0.9, \theta = 5, v^* = 100$	-12.826	29.978
$\rho = 0.9, \theta = 5, v^* = 200$	-8.703	17.212

Next, we assume that the insurer deducts the fee $g(v) = \alpha v \mathbf{1}\{v < v^*\} + \beta \alpha v \mathbf{1}\{v \geq v^*\}$. The barrier v^* above which the fee rate α is reduced with a factor β is set to

Table 4: The optimal fee rate α^* as a function of (ρ, θ, β) . The case of $g(v) = \alpha v \mathbf{1}\{v < 100\} + \beta \alpha v \mathbf{1}\{v \geq 100\}$.

The values of the parameters	The optimal fee rate
$\rho = 0, \theta = 0.01, \beta = 0.5$	0.076
$\rho = 0, \theta = 0.01, \beta = 0.33$	0.097
$\rho = 0.5, \theta = 1, \beta = 0.5$	0.125
$\rho = 0.5, \theta = 1, \beta = 0.33$	0.157
$\rho = 0.9, \theta = 5, \beta = 0.5$	0.214
$\rho = 0.9, \theta = 5, \beta = 0.33$	0.269

Table 5: The expected loss and the 95%-Value-at-Risk of the loss of the insurer's optimal hedging portfolio after paying the guarantee. The case of $g(v) = \alpha^* v \mathbf{1}\{v < 100\} + \beta \alpha^* v \mathbf{1}\{v \geq 100\}$.

The values of the parameters	The expected loss	The 95%-Value-at-Risk of the loss
$\rho = 0, \theta = 0.01, \beta = 0.5$	-0.514	16.223
$\rho = 0, \theta = 0.01, \beta = 0.33$	-0.536	16.193
$\rho = 0.5, \theta = 1, \beta = 0.5$	-1.566	16.155
$\rho = 0.5, \theta = 1, \beta = 0.33$	-1.612	15.838
$\rho = 0.9, \theta = 5, \beta = 0.5$	-9.127	18.508
$\rho = 0.9, \theta = 5, \beta = 0.33$	-9.333	19.429

the guaranteed benefit, i.e. we choose $v^* = 100$. We study how the optimal base fee rate α^* depends on the factor β and on the parameters (ρ, θ) . We also study the performance of the optimal hedging portfolio. Again, we use the pricing condition (4.4) for determining the optimal fee and the Least Squares Monte Carlo for solving the BSDE (3.4). The results are presented in Tables 4-5. Let us remark that the case with $v^* = 100$ and the case with $v^* = 200$ from Tables 2-3 correspond to the state-dependent fees $g(v) = \alpha v \mathbf{1}\{v < 100\} + \alpha \beta v \mathbf{1}\{v \geq 100\}$ with $\beta = 0$ and $\beta = 1$. It is clear that the higher the factor β is, the lower the optimal base fee rate α^* is. If we compare the results from Tables 2 and 4, then we can conclude that it might be a good strategic decision for the insurer to issue a contract under which the fee rate

is reduced (but not vanished) if the account value is above the guaranteed benefit. Even with the reduction factor $\beta = 0.33$ the optimal base fee rates α^* are acceptable from the point of view of policyholders compared to the unacceptable optimal fee rates for the contract under which the fee is not deducted if the account value is above the guaranteed benefit ($\beta = 0$) and the lowest optimal fee rates for the contract under which the proportional fee is deducted ($\beta = 1$). Other calculations confirm that introducing a reduced fee rate, which is applied if the account value is above the guaranteed benefit, does not increase the optimal base fee rate to high levels. It is worth noticing that the optimal fee rate α^* for $g(v) = \alpha v \mathbf{1}\{v < 110\}$ is close to the optimal fee rate α^* for $g(v) = \alpha v \mathbf{1}\{v < 100\} + 0.5\alpha v \mathbf{1}\{v \geq 100\}$. In Table 4 we can also observe that the stronger the dependence between the fund F and the stock S is, the higher the optimal base fee rate α^* is. This pattern agrees with the interpretation we previously discussed for the fee $g(v) = \alpha v \mathbf{1}\{v < v^*\}$. If we now look at the results from Tables 3 and 5 we can notice that the expected loss and the Value-at-Risk of the loss of the insurer's hedging portfolio under the optimal fee rate and the optimal hedging strategy increase in the factor β (except the case of the Value-at-Risk for $\rho = 0.9, \theta = 5$). This property has been also observed in other calculations. We can conclude that selling the variable annuity contract with a base fee rate and a reduced fee rate, which is applied if the account value is above the guaranteed benefit, is beneficial to the insurer from the point of view of hedging the guarantee (it improves the expected loss of the optimal hedging portfolio for all (ρ, θ) and the Value-at-Risk of the loss of the optimal hedging portfolio for small and moderate (ρ, θ) compared to the contract with a constant fee rate). The best hedging results are obtained if the insurer does not deduct a fee when the account value is above the guaranteed benefit ($\beta = 0$). The effect of state-dependent fees on hedging the guaranteed benefit must be investigated in depth and is left for future research. Finally, we would like to point out that the results from Tables 2-5 are based on one set of 10000 scenarios generated for the fund F and the stock S , hence the results are comparable.

6 Conclusions

We have studied the problem of pricing and hedging variable annuity contracts for which the fee deducted from the policyholder's account depends on the account value. To solve the pricing and hedging problem in our incomplete financial market we have applied a quadratic objective. We have derived an equation from which the fee for the guaranteed benefit can be calculated and we have characterized an optimal strategy which allows the insurer to hedge the benefit. Since we have used Backward Stochastic Differential Equations to solve the quadratic optimization problem, our results on pricing and hedging can be easily extended to include stochastic coefficients in the asset price dynamics (stochastic interest rate and stochastic volatility) and path-dependent guarantees. The extension covering mortality risk and guaranteed death benefits is left for future research.

References

- Ankirchner, S., Heine, G., 2012. Cross hedging with stochastic correlation. *Finance and Stochastics* 16, 17-43.
- Applebaum, D., 2004. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge.
- Bacinello, A.R., Millosovich, P., Olivieri, A., Pitacco, E., 2011. Variable annuities: A unifying valuation approach. *Insurance: Mathematics and Economics* 49, 285-297.
- Bauer, D., Kling, A., Russ, J., 2008. A universal pricing framework for guaranteed minimum benefits in variable annuities. *ASTIN Bulletin-Actuarial Series in Non-Life Insurance* 38, 6-21.
- Bernard, C., Hardy, M., MacKay, A., 2014. State-dependent fees for variable annuity guarantees. *ASTIN Bulletin*, in press.

- Bernard, C., MacKay, A., Muehlbeyer, M., 2014. Optimal surrender policy for variable annuity guarantees. *Insurance: Mathematics and Economics* 55, 116-128.
- Bouchard, B., Elie, R., 2008. Discrete time approximation of decoupled forward backward SDE with jumps. *Stochastic Processes and their Applications* 118, 53-75.
- Coleman, T.F., Kim, Y., Li, Y., Patron, M., 2007. Robustly hedging variable annuities with guarantees under jump and volatility risks. *Journal of Risk and Insurance* 74, 347-376.
- Cont, R., Tankov, P., 2004. *Financial Modelling with Jump Processes*. Chapman & Hall.
- Deelstra, G., Rayée, G., 2013. Pricing Variable Annuity Guarantees in a local volatility framework. *Insurance: Mathematics and Economics* 53, 650-663.
- Delong, L., 2013. *Backward Stochastic Differential Equations with Jumps and their Actuarial and Financial Applications*. Springer London.
- El Karoui, N., Peng, S., Quenez, M.C., 1997. Backward stochastic differential equations in finance. *Mathematical Finance* 7, 1-71.
- Fujii, M., Takahashi, A., 2014. Making mean-variance hedging implementable in a partially observable market. *Quantitative Finance*, in press.
- Hardy, M., 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. John Wiley & Sons.
- Jeanblanc, M., Mania, M., Santacrose, M., Schweizer, M., 2012. Mean-variance hedging via stochastic control and BSDEs for general semimartingales. *The Annals of Applied Probability* 22, 2388-2428.
- Kharroubi, I., Lim, T., Ngoupeyou, A., 2013. Mean-variance hedging on uncertain time horizon in a market with a jump. *Applied Mathematics and Optimization* 68, 413-444.

- Kohlmann, M., Xiong, D., Ye, Z., 2010. Mean-variance hedging in a general jump diffusion model. *Applied Mathematical Finance* 17, 29-57.
- Lim, A., 2005. Mean-variance hedging when there are jumps. *SIAM Journal on Control and Optimization* 44, 1893-1922.
- Longstaff, F., Schwartz, E., 2001. Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies* 14, 113-147.
- Meyer-Brandis, T., Proske, F., 2010. Construction of strong solutions of SDE's via Malliavin calculus. *Journal of Functional Analysis* 258, 3922-3953.
- Meyer-Brandis, T., Proske, F., 2006. On the existence and explicit representability of strong solutions of Lévy noise driven SDE's with irregular coefficients. *Communications in Mathematical Science* 4, 129-154.
- Øksendal, B., Hu, Y., 2008. Partial information linear quadratic control for jump diffusions. *SIAM Journal on Control and Optimization* 47, 1744-1761.
- Øksendal, B., Sulem, A., 2004. *Applied Stochastic Control of Jump Diffusions*. Springer-Verlag.
- Quittard-Pinon, F., Kelani, A., 2013. Pricing Equity Index Annuities with surrender options in four models. *Asia-Pacific Journal of Risk and Insurance* 7, 105-142.
- Schweizer, M., 1996. Approximation Pricing and the Variance-Optimal Martingale Measure. *Annals of Probability* 24, 206-236.