Mean-Variance Optimization Problems for an Accumulation Phase in a Defined Benefit Plan

Łukasz Delong[1,2,*], Russell Gerrard[2], Steven Haberman[2]

[1] *Institute of Econometrics, Division of Probabilistic Methods,*

Warsaw School of Economics, Niepodległości 162, 02-554 Warsaw, Poland

[2] *Faculty of Actuarial Science and Insurance,*

Cass Business School, 106 Bunhill Row, London EC1Y 8TZ, United Kingdom

[*] corresponding author, e-mail: lukasz.delong@sgh.waw.pl, tel/fax:(+48) 22 5648617.

Abstract

In this paper we deal with contribution rate and asset allocation strategies in a pre-retirement accumulation phase. We consider a single cohort of workers and investigate a retirement plan of a defined benefit type in which an accumulated fund is converted into a life annuity. Due to the random evolution of a mortality intensity, the future price of an annuity, and as a result, the liability of the fund, is uncertain. A manager has control over a contribution rate and an investment strategy and is concerned with covering the random claim. We consider two mean-variance optimization problems, which are quadratic control problems with an additional constrain on the expected value of the terminal surplus of the fund. This functional objectives can be related to the well-established financial theory of claim hedging. The financial market consists of a risk-free asset with a constant force of interest and a risky asset which price is driven by a Lévy noise, whereas the evolution of a mortality intensity is described by a stochastic differential equation driven by a Brownian motion. Techniques from the stochastic control theory are applied in order to find optimal strategies.

Keywords: Lévy diffusion financial market, stochastic mortality intensity process, Hamilton-Jacobi-Bellman equation, Feynman-Kac representation.

JEL: G11, G23, C61.
IM10, IE43, IB13.

1 Introduction

Pension funds have gained a lot of interest in recent years since they become large investors in financial markets and their role in providing retirement benefits substantially increased.

In this paper we investigate a retirement plan of a defined benefit type in continuous time economy. We deal with a single cohort of workers, who enter the plan and retire at the same age, and assume that the cohort is stable across an accumulation phase, which means that every member who withdraws is replaced by another at the same age. At the time of retirement an accumulated fund is used to purchase life annuities whose amounts are related to the final salaries of the participants. The total value of the salaries is known in advance and instead of dealing with the risk of labour income we focus on the risk of annuity price.

We extend the existing results from the insurance literature. We consider a financial market with an asset driven by a Lévy process and we assume that a mortality intensity follows a diffusion process. The common, in the pension fund control, mean-square error criterion is replaced by the mean-variance criterion. These extensions are not only theoretically interesting, but are also of great practical relevance. To best our knowledge the optimal control problem in this framework is taken up for the first time. The main goal of this paper is to derive the optimal investment strategy and the optimal supplementary cost in our new model. The contribution from the mathematical point, is to find classical solutions of the corresponding Hamilton-Jacobi-Bellman equations.

One of the most important characteristic of financial assets returns is their high variability, resulting from the heavy-tailed nature of empirical returns and observable large sudden movements in stock prices. The so-called six-standard deviation market moves are repeatedly seen in the financial markets around the world. This properties rule out the possibility that the marginal distribution of an asset return is Gaussian. Moreover, all models of the stock price dynamics which generate continuous sample paths are also inadequate. It is now well-known, see chapters 1 and 7 in Cont, Tankov (2005), that Lévy processes can easily reproduce heavy tails, skewness and other distributional properties of asset returns, and, what is very important as well, can generate discontinuities in the price dynamics. As Lévy processes generate more realistic sample paths of stock prices, one should replace, in the celebrated Black-Scholes model, a Gaussian noise by a Lévy noise. This extension is worth the effort, as a "good model" of investment returns is a crucial factor when constructing strategies for an accumulation phase of a retirement plan.

In 1980's and 1990's the mortality improvements turned out to be much greater than forecasts and the unexpected decrease in mortality rates effected the solvency of pension providers. Over the last 20 years the mortality improvements varied substantially and mortality rates were evolving in a random fashion, see Cairns *et al* (2004b). One can notice the general trend but there is still an unpredictable factor left which cannot be handled by any deterministic model. This is the reason why probabilistic models of the mortality evolution have appeared in the literature, see Dahl (2004), Luciano, Vigna (2005), Schrager (2006), Ballotta, Haberman (2006). We follow this probabilistic approach to mortality modelling and we consider the annuity price as a random variable whose randomness arises due to the unpredictable evolution of a mortality intensity. Clearly, the level of a mortality intensity is one of the two important factors which influences the annuity price. The second factor, which is the level of the interest rate, is investigated in the paper of Cairns *et al* (2004a) in a framework of a defined contribution pension plan.

There are a lot of papers dealing with defined benefit pension plans, see for example Haberman, Sung (1994), Haberman *et al* (2000), Cairns (2000), Chang *et al* (2003), Haberman, Sung (2005). However, we are aware of only two in which a pension liability is a random variable. In Josa-Fombellida, Rincón-Zapatero (2004) an aggregate pension plan in infinite time is considered and benefits are modelled as a geometric Brownian motion, which introduces another state variable in a control problem. A similar optimization model is investigated in the recent paper of Ngwira, Gerrard (2006), in which an asset return follows an exponential, jump-diffusion, Lévy process with lognormally distributed jumps (the so-called Merton asset model). The main contribution of this work is the analysis of the effect of a jump magnitude on an asset allocation strategy.

A plan manager faces the task of accumulating enough funds to cover the price of the required annuities. This problem can be viewed, from the point of mathematical finance, as a hedging problem of a liability. More precisely, it is the hedging problem in an incomplete financial market where there are two " risky assets", one of which is tradeable, while the contingent claim is a function of the second, non-tradeable instrument. This kind of problem, in a more general setting of correlated assets and contingent claims, was introduced by Duffie, Richardson (1991) and Schweizer (1992), and solved by applying martingale techniques and projection methods. Recently, this issue has been taken up in Hipp, Taksar (2005) and stochastic control techniques are applied to minimize a hedging error in $p-$norm. Explicit solutions to Hamilton-Jacobi-Bellman equations are only derived in the cases of constant liabilities. In Kohlmann, Peisl (2000) techniques of Backward Stochastic Differential Equations are applied to solve the problem. The explicit result is given for some simple case.

All above papers deal with the most common criterion of minimization of a mean-square error, while our objective is to minimize variance given the fixed expected value of the terminal surplus. A mean-variance problem was introduced in Markowitz (1952) and this work has laid down foundations of the modern financial theory. In the financial literature the mean-variance objective is usually applied to solve portfolio selection problems for self-financing wealth processes. We refer the reader to Zhou, Li (2000), where the original mean-variance problem is embedded into an auxiliary problem, which is then solved by stochastic control methods, as well as to Bielecki *et al* (2005) and Lim (2004), where techniques of Backward Stochastic Differential Equations are used in the presence of random market coefficients. Recently, in Bielecki *et al* (2004) the mean-variance approach is applied to hedge general claims. Some results, based on orthogonal projection methods, are derived for defaultable claims. We would like to point out that in all mentioned papers, except in Ngwira, Gerrard (2006), the diffusion dynamics of a risky asset's price is assumed.

This paper is structured as follows. In section 2 we introduce a financial market, a stochastic mortality intensity process, a retirement plan and its liability. Our problems are formulated in section 3. A mean-variance optimization problem is solved in section 4, whereas a generalized mean-variance problem with a running cost of supplementary contribution rates is considered in section 5. We also investigate some numerical examples. Summarizing comments are stated in section 6.

2 The model

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T'}$ for some finite T' which denotes the maximum future life-time of the plan's participants. The filtration satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). The filtration $\mathbb{F}$ consists of two subfiltrations: we set $\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M$, where $\mathbb{F}^F$ contains information about the financial market and $\mathbb{F}^M$ contains information about the mortality intensity. We assume that the subfiltrations $\mathbb{F}^F$ and $\mathbb{F}^M$ are independent, which means that the dynamics of the price of a risky asset is independent of a mortality intensity. The measure $\mathbb{P}$ is the real-world, objective probability measure. All expected values are taken with respect to measure $\mathbb{P}$ and the conditional expected value $\mathbb{E}^{\mathbb{P}}[\cdot|X(t) = x, \lambda(t) = \lambda]$ is denoted as $\mathbb{E}^{t,x,\lambda}[\cdot]$. The class of $\mathcal{C}^{1,2,2}([0,T) \times \mathbb{R} \times (0,\infty)) \cap \mathcal{C}([0,T] \times \mathbb{R} \times (0,\infty))$ functions is denoted simply by $\mathcal{C}$.

In the following subsections we introduce a financial market, a stochastic mortality intensity process and a defined benefit plan.

2.1 The financial market

We consider a Lévy diffusion version of a Black-Scholes financial market. The price of a risk-free asset $S_0 := (S_0(t), 0 \leq t \leq T')$ is described by the ordinary differential equation

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad S_0(0) = 1, \tag{2.1}$$

where r denotes a rate of interest. The second tradeable financial instrument in the market is a risky stock and the dynamics of its price $S := (S(t), 0 \leq t \leq T')$ are given by the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + \xi dL(t), \quad S(0) = 1, \tag{2.2}$$

where μ and ξ denote a drift and a volatility, and $L := (L(t), 0 \leq t \leq T')$ denotes a zero-mean Lévy process (a process with independent and stationary increments), $\mathbb{F}^F$-adapted with càdlàg sample paths (paths which are continuous on the right and have limits on the left).

The zero-mean Lévy process L is assumed to satisfy the following Lévy-Itô decomposition, see chapter 2.4 in Applebaum (2004),

$$L(t) = \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z\big(M(ds \times dz) - \nu(dz)ds\big), \tag{2.3}$$

where $W := (W(t), 0 \leq t \leq T')$ is a Brownian motion and $M((s,t] \times A) = \#\{s < u \leq t : (L(u) - L(u-)) \in A\}$ is a Poisson random measure, independent of W, with a deterministic, time-homogeneous intensity measure $\nu(dz)dt$. The Poisson random measure counts the number of jumps of a particular size in the given time interval. The measure ν is called a Lévy measure and verifies $\int_{\mathbb{R}}(z^2 \wedge 1)\nu(dz) < \infty$. Let us recall that the compensated measure $\tilde{M}((0,t] \times A) = M((0,t] \times A) - \nu(A)t$ is a martingale-valued measure, that is $\tilde{M} := (\tilde{M}((0,t] \times A), 0 \leq t \leq T)$ is a $\mathbb{F}^F$-martingale for all all Borel sets $A \in \mathcal{B}(\mathbb{R} - \{0\})$. We refer the reader to Applebaum (2004) for mathematical details concerning Lévy processes and Poisson random measures.

We make the following assumptions concerning the coefficients and the intensity measure:

(A1) r, μ, σ are non-negative constants and $r < \mu$,

(A2) we set $\xi \equiv 1$, this is no loss of generality as the process ξL has also independent and stationary increments and satisfies the Lévy-Itô decomposition,

(A3) ν is a Lévy measure on $(-1, \infty)$, such that $\nu(\{0\}) = 0$ and $\int_{z \geq 1} z^4 \nu(dz) < \infty$.

The condition **(A1)** is not necessary from the mathematical point, but it is reasonable from the economic point. It simplifies the interpretation of the derived optimal strategies. The moment condition in **(A3)** ensures that $\sup_{t\in[0,T']}\mathbb{E}[|L(t)|^4]<\infty$.

The stochastic differential equation (2.2) has a unique, positive and almost surely finite solution, given explicitly by Doléans-Dade exponential

$$\begin{aligned} S(t) &= \exp\big\{\big(\mu-\frac{1}{2}\sigma^2+\int_{z>-1}(\log(1+z)-z)\nu(dz)\big)t \\ &\quad +\sigma W(t)+\int_{(0,t]}\int_{z>-1}\log(1+z)\tilde{M}(ds\times dz)\big\} \\ &= \exp\big\{\mu_E t+\sigma W(t)+\int_{(0,t]}\int_{\mathbb{R}} z\tilde{M}_E(ds\times dz)\big\} \end{aligned} \tag{2.4}$$

which is an exponential Lévy process with $\nu_E(A)=\nu(\{z:\log(1+z)\in A\})$, see propositions 8.21 and 8.22 in Cont, Tankov (2005). The measure ν_E should satisfy the equivalent condition **(A3')**:

(A3') ν_E is a Lévy measure on $\mathbb{R}$, such that $\nu_E(\{0\})=0$ and $\int_{z\geq 1}e^{4z}\nu_E(dz)<\infty$.

There is one to one correspondence between the measures and stock price models (2.2) and (2.4). Notice that our financial model is general enough to include not only jump-diffusion processes, like Merton or Kou models, but we can also work with infinite activity Lévy processes. We point out that in chapter 3 in Kyprianou *et al* (2005) the intensity measures ν_E for Variance Gamma and CGMY processes are estimated for five world index markets and in each case the estimated measure satisfies **(A3')**. We refer the reader to Cont, Tankov (2005) and Kyprianou *et al* (2005) in which different aspects of financial modelling with Lévy diffusion processes are investigated.

2.2 The stochastic mortality intensity

We consider the mortality intensity, $\Lambda:=(\lambda(t),0\leq t\leq T')$, as a stochastic process with the dynamics given by the stochastic differential equation

$$d\lambda(t)=\theta(t,\lambda(t))dt+\eta(t,\lambda(t))d\bar{W}(t),\quad \lambda(0)=\lambda, \tag{2.5}$$

where $\bar{W}:=(\bar{W}(t),0\leq t\leq T')$ is an $\mathbb{F}^M$-adapted Brownian motion independent of W. We assume that the process Λ is $\mathbb{F}^M$-adapted, which means that at each point of time it is possible to estimate the "true" level of the mortality intensity and use this estimate in the decision-making process. We adopt the notation that $\lambda(t)$ denotes the level of the mortality intensity in the considered cohort of participants t years after they entered a plan.

We make the following assumptions concerning the mortality intensity process:

(B1) $\theta : [0, T'] \times (0, \infty) \to \mathbb{R}, \eta : [0, T'] \times (0, \infty) \to (0, \infty)$ are continuous functions, locally Lipschitz continuous in λ, uniformly in t,

(B2) there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of bounded domains with $\bar{E}_n \subseteq (0, \infty)$ and $\bigcup_{n \geq 1} E_n = (0, \infty)$, such that the functions $\theta(t, \lambda)$ and $\eta(t, \lambda)$ are uniformly Lipschitz continuous on $[0, T'] \times \bar{E}_n$,

(B3) $\mathbb{P}(\forall_{s \in [t, T']} \lambda(s) \in (0, \infty) | \lambda(t) = \lambda) = 1$ and for all starting points $(t, \lambda) \in [0, T'] \times (0, \infty)$.

The diffusion dynamics of the intensity seems to be very reasonable, as changes in mortality occurs slowly and without sudden jumps. We would like to point out that the mortality intensity models appearing in the literature, see Dahl (2004), Luciano, Vigna (2005), Schrager (2006), Ballotta, Haberman (2006). satisfy **(B1)-(B3)** and arise as the special case of (2.5).

Under the assumptions **(B1)** and **(B3)**, for each starting point $(t, \lambda) \in [0, T'] \times (0, \infty)$, the mortality intensity process is nonexplosive on $[t, T']$ and there exists a unique strong solution to the stochastic differential equation (2.5), such that the mapping $(t, \lambda, s) \to \lambda^{t,\lambda}(s)$ is $\mathbb{P}$-a.s. continuous, see Heath, Schweizer (2000) and Becherer, Schweizer (2005). The assumption **(B2)**, together with **(A3)**/**(A3')**, is required in order to show the smoothness of the candidate value function and to prove the optimality of the derived strategy.

2.3 The retirement plan and its random liability

During the accumulation phase $[0, T]$, where $T < T'$ denotes the common time of retirement of all participants, a sponsor contributes to the plan and the funds are invested in the financial market (2.1)- (2.2). Let $X(t)$ denote the value of the accumulated fund at time t. The dynamics of the process $X^{\pi,c} := (X^{\pi,c}(t), 0 \leq t \leq T)$ are given by the stochastic differential equation

$$\begin{aligned} dX^{\pi,c}(t) &= \pi(t)\big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt \times dz)\big) \\ &\quad + \big(X^{\pi,c}(t-) - \pi(t)\big) r dt + c(t) dt, \quad X(0) = x, \end{aligned} \tag{2.6}$$

where $\pi(t)$ denotes the amount of the fund invested in the risky asset and $c(t)$ denotes the contribution rate at time t.

We assume that the market price of the annuity is calculated as the expected present value of future payments discounted with the risk-free rate, conditioned on the given level of the mortality intensity. The liability of the plan at time T is then equal to

$$Da(\lambda) = D\mathbb{E}^{T,\lambda}\Big[\int_T^{T'} e^{-r(s-T)} e^{-\int_T^s \lambda(u) du} ds\Big], \tag{2.7}$$

where D denotes the aggregate promised amount and we ignore the residual probability of payments after T', which is set in our model to zero. Consider the function $a : [0, T] \times (0, \infty) \to (0, \infty)$ defied as

$$
\begin{aligned}
a(t, \lambda) &= \mathbb{E}^{t,\lambda}[a(\lambda(T))] \\
&= \mathbb{E}[\int_T^{T'} e^{-r(s-T)} e^{-\int_T^s \lambda^{t,\lambda}(u)du} ds], \qquad (2.8)
\end{aligned}
$$

where the second equality follows from the Markov property of the mortality intensity and the law of iterated expectations. We remark that $Da(t, \lambda)$ is the expected liability of the plan at time $t \in [0, T]$ given the level of the mortality intensity at that time. Notice that the mapping $(t, \lambda) \mapsto a(t, \lambda)$ is continuous, and we can conclude, based on the theorem 1 from Heath, Schweizer (2000), that $a \in \mathcal{C}$ is the unique solution of the partial differential equation

$$
\frac{\partial a}{\partial t}(t, \lambda) + \theta(t, \lambda)\frac{\partial a}{\partial \lambda}(t, \lambda) + \frac{1}{2}\eta^2(t, \lambda)\frac{\partial^2 a}{\partial \lambda^2}(t, \lambda) = 0, \qquad a(T, \lambda) = a(\lambda). \qquad (2.9)
$$

Example 2.1. In the numerical example we consider a member who joins the plan at the age of 45 and retires at the age of 65. We assume that the mortality intensity follows an exponential Ornstein-Uhlenbeck process of the form

$$
\lambda(t) = 0,0025e^{0,08t+0,1Y(t)}, \quad dY(t) = -0,2Y(t) + d\bar{W}(t). \qquad (2.10)
$$

An exponential Ornstein-Uhlenbeck process, as a process describing the evolution of the mortality intensity over time, is applied for example in Ballotta, Haberman (2006).

Below, in table 1, we give the prices of the annuity depending on the values of the mortality intensity at time $T = 20$. The maximum future life-time is taken to be 100 years, $T' = 55$, and the discount rate is set to $r = 0,05$. The price of the annuity calculated based on deterministic mortality intensity $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$ is equal to 11,901.

It is rather clear that randomness of the mortality intensity should be taken into account. The observed differences in the annuity prices can have a significant impact on the aggregate liability of the plan. □

We assume that the contribution rate which funds the liability combines two elements: a normal cost and a supplementary cost amortizing the unfunded liability. Let $F(t)$ denote a distribution function according to which the liability (2.7) is accumulated, it denotes a percentage of the value of the benefit (2.7) accumulated during the first t years. We need the following assumption

(C) $F : [0, T] \to [0, 1]$ is absolutely continuous with respect to Lebesgue measure, and its density $f(t)$ is Lipschitz continuous on $[0, T]$.

Table 1: Prices of the annuity

λ	$Pr(\lambda(20) \leq \lambda)$	$a(\lambda)$
0,007	0,0002	12,2616
0,008	0,0028	12,1937
0,009	0,0216	12,1199
0,01	0,0878	12,0460
0,011	0,2265	11,9908
0,012	0,4212	11,9463
0,013	0,6211	11,8893
0,014	0,7817	11,8227
0,015	0,8879	11,7766
0,016	0,9478	11,7290
0,017	0,9777	11,6996
0,018	0,9911	11,6221
0,019	0,9967	11,6098
0,02	0,9988	11,5474
0,021	0,9996	11,5043

The normal cost and the actuarial liability can be defined as in Josa-Fombellida, Rincón-Zapatero (2004)

$$NC(t,\lambda) = e^{-\rho(T-t)} f(t)\mathbb{E}^{t,\lambda}[Da(\lambda(T))], \qquad 0 \leq t \leq T, \tag{2.11}$$

$$AL(t,\lambda) = e^{-\rho(T-t)} F(t)\mathbb{E}^{t,\lambda}[Da(\lambda(T))], \qquad 0 \leq t \leq T, \tag{2.12}$$

where ρ is a plan's valuation rate. The supplementary contribution rate can be a control variable, to be determined, or can be set as

$$u(t,\lambda) = \kappa\big(AL(t,\lambda) - X^{\pi}(t-)\big), \qquad 0 < t \leq T, \tag{2.13}$$

which would reflect the spread method of the fund amortization, where κ is some predefined constant and the control process $(X^{\pi}, 0 \leq t \leq T)$ depends now only on an investment strategy π.

3 Problem formulation

The aim of the manager is to manage the fund in order to cover the liability $Da(\lambda(T))$ at time T. This problem can be viewed as a hedging problem. Clearly, the liability is $\mathcal{F}_T^M$-measurable random variable and it is not attainable as it cannot be replicated by investing in the financial market (2.1)-(2.2). The question arises how to quantify the risk of not covering the claim and how to minimize the chosen functional to arrive at optimal controls. One of the well-established methods of hedging in incomplete financial markets is a quadratic hedging which was introduced in Föllmer, Sondermann (1986) in a martingale case and then extended in Schweizer (1996) with the concept of a minimal variance martingale measure.

Let us first assume that the supplementary contribution rate is set according to (2.13). Under the quadratic hedging approach the manager should construct the investment portfolio in order to minimize the hedging error at maturity T in a mean square sense

$$\inf_{\pi} \mathbb{E}\big[X^{\pi}(T) - Da(\lambda(T))\big]^2. \tag{3.1}$$

Such an objective is quite reasonable as the main goal of any retirement plan should be its safety and stability. However, more intuitive is to minimize the variance of the surplus at maturity T

$$\inf_{\pi} \mathrm{Var}\big[X^{\pi}(T) - Da(\lambda(T))\big]. \tag{3.2}$$

In fact we would like to solve the problem in which the variance of the surplus is minimized given the fixed expected value of the surplus. In this paper we deal with the following constrained quadratic control problem

$$\begin{cases} \inf_{\pi} \mathrm{Var}\big[X^{\pi}(T) - Da(\lambda(T))\big] \\ \mathbb{E}\big[X^{\pi}(T) - Da(\lambda(T))\big] = 0 \end{cases} \tag{3.3}$$

We can also add the supplementary contribution rate into our optimization problem and minimize the variance of the surplus along with the expected value of the squares of future supplementary costs which leads to the generalized version of (3.3)

$$\begin{cases} \inf_{\pi,u} \mathbb{E}\big[\int_0^T u^2(t)dt\big] + \alpha \mathrm{Var}\big[X^{\pi,u}(T) - Da(\lambda(T))\big] \\ \mathbb{E}\big[X^{\pi,u}(T) - Da(\lambda(T))\big] = 0 \end{cases}, \tag{3.4}$$

where $\alpha > 0$ attaches a weight to the terminal variance with respect to the running cost of contributions. Without difficulty we can equate the expected value of the surplus in (3.3) and (3.4) to some constant K or just load the benefit D.

It is well-recognized that the goal of the pension manager should be to minimize the solvency risk and the contribution risk. The common approach in the defined benefit pension literature is to measure the risk by means of a quadratic objective, see Haberman, Sung (1994), Haberman *et al* (2000), Cairns (2000), Chang *et al* (2003), Josa-Fombellida, Rincón-Zapatero (2004), Haberman, Sung (2005), Ngwira, Gerrard (2006). In all this papers the mean-square error objective is applied, whereas we are interested in the mean-variance objective.

4 Mean-variance optimization problem

In this section we solve the mean-variance hedging problem (3.3). The variance optimization criterion can be handled by incorporating the equality constraint on the expected value of the surplus into the objective function by using Lagrange multiplier. Instead of dealing with (3.3) we can first solve the following stochastic control problem

$$\inf_{\pi\in\mathcal{A}} \mathbb{E}^{0,x,\lambda}\big[\big(X^{\pi}(T)-Da(\lambda(T))\big)^2-\beta\big(X^{\pi}(T)-Da(\lambda(T))\big)\big] \tag{4.1}$$

and then choose a Lagrange multiplier β such that the constraint on the expected value of the terminal surplus is satisfied

$$\mathbb{E}^{0,x,\lambda}\big[X^{\hat{\pi},\beta}(T)-Da(\lambda(T))\big]=0, \tag{4.2}$$

where $\hat{\pi}$ is the optimal strategy for (4.1).

Let us introduce the set of admissible strategies and two operators.

Definition 4.1. *A strategy $(\pi(t), 0<t\le T)$ is admissible, $\pi\in\mathcal{A}$, if it satisfies the following conditions:*

1. *$\pi:(0,T]\times\Omega\to\mathbb{R}$ is a predictable mapping with respect to filtration $\mathbb{F}$,*
2. *$\mathbb{E}^{0,x,\lambda}[\int_0^T \pi^2(t)dt]<\infty$,*
3. *the stochastic differential equation (2.6) has a unique solution X^{π} on $[0,T]$.*

It is well-known that it is sufficient to consider only Markov strategies, see chapter 3 in Øksendal, Sulem (2005). We point out that for any $\pi\in\mathcal{A}$ the process X^{π}, which satisfies (2.6), is a square integrable semimartingale with cádlág sample paths, see chapter 4.3.3 in Applebaum (2004).

Definition 4.2. *The integro-differential operator $\mathcal{L}_F$ is given by*

$$\begin{aligned}\mathcal{L}_F^{\pi}\phi(t,x,\lambda) &= \big(\pi(\mu-r)+xr+NC(t,\lambda)+k(AL(t,\lambda)-x)\big)\frac{\partial\phi}{\partial x}(t,x)\\ &+ \frac{1}{2}\pi^2\sigma^2\frac{\partial^2\phi}{\partial x^2}(t,x)\\ &+ \int_{z>-1}\big(\phi(t,x+\pi z)-\phi(t,x)-\pi z\frac{\partial\phi}{\partial x}(t,x)\big)\nu(dz),\end{aligned} \tag{4.3}$$

whereas the differential operator $\mathcal{L}_M$ is given by

$$\mathcal{L}_M\phi(t,\lambda)=\theta(t,\lambda)\frac{\partial\phi}{\partial\lambda}(t,\lambda)+\frac{1}{2}\eta^2(t,\lambda)\frac{\partial^2\phi}{\partial\lambda^2}(t,\lambda). \tag{4.4}$$

This operators are defined for all functions ϕ such that the partial derivatives and the integral in (4.3) and (4.4) exist pointwise.

Let us introduce the optimal value function for the problem (4.1)

$$V(t,x,\lambda)=\inf_{\pi\in\mathcal{A}}\mathbb{E}^{t,x,\lambda}\big[\big(X^{\pi}(T)-Da(\lambda(T))\big)^2-\beta\big(X^{\pi}(T)-Da(\lambda(T))\big)\big], \quad (4.5)$$

We can prove the following classical verification theorem.

Theorem 4.1. *Let $v\in\mathcal{C}$ satisfy for every $\pi\in\mathcal{A}$*

$$0\leq\frac{\partial v}{\partial t}(t,x,\lambda)+\mathcal{L}_F^{\pi}v(t,x,\lambda)+\mathcal{L}_Mv(t,x,\lambda), \quad (4.6)$$

for all $(t,x,\lambda)\in[0,T)\times\mathbb{R}\times(0,\infty)$, with

$$v(T,x,\lambda)=\big(x-Da(\lambda)\big)^2-\beta\big(x-Da(\lambda)\big),\quad \forall(x,\lambda)\in\mathbb{R}\times(0,\infty). \quad (4.7)$$

Assume also that for every $\pi\in\mathcal{A}$

$$\mathbb{E}^{0,x,\lambda}\Big[\int_0^T\int_{z>-1}|v(t,X^{\pi}(t-)+\pi(t)z,\lambda(t))-v(t,X^{\pi}(t-),\lambda(t))|^2\nu(dz)dt\Big]<\infty, \quad (4.8)$$

$$\mathbb{E}^{0,x,\lambda}\Big[\int_0^T\int_{z>-1}|v(t,X^{\pi}(t-)+\pi(t)z,\lambda(t))-v(t,X^{\pi}(t-),\lambda(t)) -\pi(t)z\frac{\partial v}{\partial x}(t,X^{\pi}(t-),\lambda(t))|\nu(dz)dt\Big]<\infty, \quad (4.9)$$

and

$\{v^+(\tau,X^{\pi}(\tau),\lambda(\tau))\}_{0<\tau\leq T}$ *is uniformly integrable for all $\mathbb{F}$-stopping times τ.* (4.10)

Then

$$v(t,x,\lambda)\leq V(t,x,\lambda),\quad\forall(t,x,\lambda)\in[0,T]\times\mathbb{R}\times(0,\infty). \quad (4.11)$$

Moreover, if there exists an admissible control $\hat{\pi}\in\mathcal{A}$ such that

$$0=\frac{\partial v}{\partial t}(t,x,\lambda)+\mathcal{L}_F^{\hat{\pi}}v(t,x,\lambda)+\mathcal{L}_Mv(t,x,\lambda) \quad (4.12)$$

holds for all $(t,x,\lambda)\in[0,T)\times\mathbb{R}\times(0,\infty)$, and

$\{v(\tau,X^{\hat{\pi}}(\tau),\lambda(\tau))\}_{0<\tau\leq T}$ *is uniformly integrable for all $\mathbb{F}$-stopping times τ,* (4.13)

then

$$v(t,x,\lambda)=V(t,x,\lambda),\quad\forall(t,x,\lambda)\in[0,T]\times\mathbb{R}\times(0,\infty), \quad (4.14)$$

and $\hat{\pi}$ is the optimal strategy for the problem (4.1).

Proof:
For each $(t,x,\lambda) \in [0,T)\times\mathbb{R}\times(0,\infty)$ define a sequence of stopping times $t_n = \inf\{s \in (t,T]; |X^{\pi}(s) - x| + |\lambda(s) - \lambda| > n\}$, and choose ϵ such that $0 < \epsilon < T - t$. Then apply Itô formula for semimartingales, see theorem 4.4.7 in Applebaum (2004), to the function v on the time interval $[t, t_n \wedge (T-\epsilon)]$. The expected values of the stochastic integrals with respect to the Brownian motion and the compensated Poisson measure are equal to zero due to the localizing sequence and the condition (4.8) The next steps are rather standard and we refer the reader to theorem 3.1 in Øksendal, Sulem (2005) for details. Taking the limit $n \to \infty, \epsilon \to 0$, the inequality (4.11) and the equality (4.14) are established with Fatou's lemma and Lebesgue's dominated convergence theorem. □

We point out that our localization procedure is crucial as we can omit some of the conditions stated in theorem 3.1 in Øksendal, Sulem (2005). Without this localization procedure it would be hard, and would require stronger assumptions, to check the general conditions of the verification theorem and verify that our solution is optimal.

As our optimization problem (4.1) is quadratic it is natural to try to find a solution in the form $v(t,x,\lambda) = A(t,\lambda)x^2 + B(t,\lambda)x + C(t,\lambda)$. However, we would like to point out that finding the solution in this form seems to be a novelty in the stochastic control theory, see also Delong, Gerrard (2007). With this choice of the value function the optimal strategy $\hat{\pi}$ which minimizes the right hand side of (4.6) is given by

$$\hat{\pi}(t,x,\lambda) = -\bar{\delta}\big(x + \frac{B(t,\lambda)}{2A(t,\lambda)}\big), \quad \bar{\delta} = \frac{\mu - r}{\sigma^2 + \int_{z>-1} z^2\nu(dz)}. \tag{4.15}$$

Substituting (4.15) into (4.12) and collecting the terms we arrive at three partial differential equations

$$\begin{cases} 0 = \frac{\partial A}{\partial t}(t,\lambda) + \mathcal{L}_M A(t,\lambda) + (2r - 2\kappa - \delta)A(t,\lambda), \\ A(T,\lambda) = 1, \end{cases} \tag{4.16}$$

$$\begin{cases} 0 = \frac{\partial B}{\partial t}(t,\lambda) + \mathcal{L}_M B(t,\lambda) + (r - \kappa - \delta)B(t,\lambda) \\ +2A(t,\lambda)(NC(t,\lambda) + \kappa AL(t,\lambda)), \\ B(T,\lambda) = -\beta - 2Da(\lambda), \end{cases} \tag{4.17}$$

$$\begin{cases} 0 = \frac{\partial C}{\partial t}(t,\lambda) + \mathcal{L}_M C(t,\lambda) + B(t,\lambda)(NC(t,\lambda) + \kappa AL(t,\lambda)) - \frac{B^2(t,\lambda)}{4A(t,\lambda)}\delta, \\ C(T,\lambda) = D^2a^2(\lambda) + \beta Da(\lambda), \end{cases} \tag{4.18}$$

where $\delta = \bar{\delta}(\mu - r)$.

A solution to (4.16) can be stated explicitly by noticing that the time-dependent function of the form

$$A(t) = e^{(2r-2\kappa-\delta)(T-t)} \tag{4.19}$$

satisfies (4.16). As far as the next two partial differential equations are concerned we can prove the following lemma.

Lemma 4.1. *There exist unique, uniformly bounded solutions in class $\mathcal{C}$ to the partial differential equations (4.17) and (4.18). In particular, the unique solution to (4.17) satisfies the Feynmam-Kac formula*

$$\begin{aligned} B(t,\lambda) &= \mathbb{E}^{t,\lambda}\big[-\big(\beta+2Da(\lambda(T))\big)e^{(r-\kappa-\delta)(T-t)} \\ &\quad+\int_t^T 2A(s)\big(NC(s,\lambda(s))+\kappa AL(s,\lambda(s))\big)e^{(r-\kappa-\delta)(s-t)}ds\big]. \end{aligned} \tag{4.20}$$

Proof:

We follow the proof of proposition 2.3 in Becherer, Schweizer (2005). Choose $\epsilon > 0$ and consider the partial differential equation (4.17) on the time interval $[0, T-\epsilon]$. Notice that the functions NC and AL are uniformly Hölder continuous on compact subsets of $[0, T-\epsilon] \times \bar{E}_n$, due to the assumption **(C)** and the smoothness of the function a. Applying Lebesgue's dominated convergence theorem one can easily show that the mapping $(t, \lambda) \mapsto B(t, \lambda)$, defined in (4.20), is continuous. Based on the theorem 1 in Heath, Schweizer (2000) we can conclude that the function

$$\begin{aligned} B(t,\lambda) &= \mathbb{E}^{t,\lambda}\big[B(T-\epsilon,\lambda(T-\epsilon))e^{(r-\kappa-\delta)(T-\epsilon-t)} \\ &\quad+\int_t^{T-\epsilon} 2A(s)\big(NC(s,\lambda(s))+\kappa AL(s,\lambda(s))\big)e^{(r-\kappa-\delta)(s-t)}ds\big], \end{aligned} \tag{4.21}$$

is the unique classical solution to the equation (4.17) on the time interval $[0, T-\epsilon]$. As ϵ is arbitrary, the existence of the classical solution B on $[0, T] \times (0, \infty)$ follows. The uniform boundness is obvious.

The result concerning the partial differential equation (4.18) is proved analogously. □

By substituting (2.11), (2.12), (4.19), applying Fubini's theorem, the Markov property of the mortality intensity and the law of iterated expectations we can arrive at

$$\begin{aligned} B(t,\lambda) &= -\beta e^{(r-\kappa-\delta)(T-t)}+\big(-2e^{(r-\kappa-\delta)(T-t)}+2e^{(r-\kappa-\rho)T+(r-\kappa-\delta)(T-t)} \\ &\quad\times\int_t^T e^{(\kappa+\rho-r)s}\big(f(s)+\kappa F(s)\big)ds\big)\mathbb{E}^{t,\lambda}[Da(\lambda(T))]. \end{aligned} \tag{4.22}$$

Moreover, integration by parts

$$\begin{aligned}
&\int_t^T e^{(\kappa+\rho-r)s}(f(s)+\kappa F(s))ds \\
&= e^{(\kappa+\rho-r)T} - e^{(\kappa+\rho-r)t}F(t) - (\rho-r)\int_t^T e^{(\kappa+\rho-r)s}F(s)ds \\
&= \frac{\kappa}{\kappa+\rho-r}(e^{(\kappa+\rho-r)T} - e^{(\kappa+\rho-r)t}F(t)) \\
&\quad +\frac{\rho-r}{\kappa+\rho-r}\int_t^T e^{(\kappa+\rho-r)s}f(s)ds, \qquad (4.23)
\end{aligned}$$

yields

$$\begin{aligned}
B(t,\lambda) &= -\beta e^{(r-\kappa-\delta)(T-t)} - 2e^{(r-\kappa-\delta)(T-t)}\big(e^{(r-\kappa-\rho)(T-t)}F(t) \\
&\quad +(\rho-r)e^{(r-\kappa-\rho)T}\int_t^T e^{(k+\rho-r)s}F(s)ds\big)\mathbb{E}^{t,\lambda}[Da(\lambda(T))] \\
&= -\beta e^{(r-\kappa-\delta)(T-t)} - 2e^{(r-\kappa-\delta)(T-t)}\big(\frac{\rho-r}{\kappa+\rho-r} + \frac{\kappa}{\kappa+\rho-r}e^{(r-\kappa-\rho)(T-t)}F(t) \\
&\quad -\frac{\rho-r}{\kappa+\rho-r}e^{(r-\kappa-\rho)T}\int_t^T e^{(\kappa+\rho-r)s}f(s)ds\big)\mathbb{E}^{t,\lambda}[Da(\lambda(T))]. \qquad (4.24)
\end{aligned}$$

Notice that B is non-positive in the case of $\rho \geq r$ and $\beta \geq 0$. We would like to point out that the choice of the plan's valuation rate $\rho \geq r$ is very reasonable.

Let us now investigate the fund process $X^{\hat{\pi}}$ under the optimal strategy. Its dynamics are given by the stochastic differential equation

$$\begin{aligned}
dX^{\hat{\pi}}(t) &= \big\{-\delta\big(X^{\hat{\pi}}(t-) + \frac{B(t,\lambda(t))}{2A(t)}\big) + X^{\hat{\pi}}(t-)rdt \\
&\quad +NC(t,\lambda(t)) + \kappa AL(t,\lambda(t)) - \kappa X^{\hat{\pi}}(t-)\big\}dt \\
&\quad -\bar{\delta}\big(X^{\hat{\pi}}(t-) + \frac{B(t,\lambda(t))}{2A(t)}\big)\big(\sigma dW(t) + \int_{z>-1} z\tilde{M}(dt\times dz)\big), \quad (4.25)
\end{aligned}$$

with the initial condition $X(0)=x$. We can prove the following lemma.

Lemma 4.2. *The stochastic differential equation (4.25), given the initial condition $X(0)=x\in\mathbb{R}$, has a unique solution on $[0,T]$ in the space of semimartingales processes with cádlág sample paths. This solution has finite fourth moment $\sup_{t\in[0,T]}\mathbb{E}^{0,x,\lambda}[|X^{\hat{\pi}}(t)|^4]<\infty$.*

Proof:

The existence and uniqueness follow from the general theory of stochastic differential equations driven by discontinuous semimartingales in the case of functional Lipschitz coefficients, see theorem V.7 in Protter (2005). To arrive at the second part of the lemma, one should define the sequence of stopping times $\tau_m = \inf\{s \in (0,T], |X^{\hat{\pi}}(s) - x| > m\}$ and show that

$$\mathbb{E}^{0,x,\lambda}\big[|X^{\hat{\pi}}(t)|^4 \mathbf{1}\{\tau_m > t\}\big] \leq K\big(1 + \int_0^t \mathbb{E}^{0,x,\lambda}\big[|X^{\hat{\pi}}(s)|^4 \mathbf{1}\{\tau_m > s\}\big] ds\big), \quad (4.26)$$

for arbitrary $(t, x, \lambda) \in [0,T] \times \mathbb{R} \times (0, \infty)$ and some constant $K < \infty$. This can be obtained by applying an estimate of moments of stochastic integrals of predictable processes with respect to Lévy processes, see Protter (2005). The result follows from Gromwell inequality and Fatou's lemma by taking the limit as $m \to \infty$. □

We proceed now to find the value of the Lagrange multiplier. The Itô differential (4.25) can be rewritten in the integral form

$$\begin{aligned} X^{\hat{\pi}}(t) &= x + \int_0^t \big\{ -\delta\big(X^{\hat{\pi}}(s-) + \frac{B(s,\lambda(s))}{2A(s)}\big) + X^{\hat{\pi}}(s-) r \\ &\quad + NC(s,\lambda(s)) + \kappa AL(s,\lambda(s)) - \kappa X^{\hat{\pi}}(s-)\big\} ds \\ &\quad - \int_0^t \bar{\delta}\big(X^{\hat{\pi}}(s-) + \frac{B(s,\lambda(s))}{2A(s)}\big)\big(\sigma dW(s) + \int_{z>-1} z\tilde{M}(ds \times dz)\big). \end{aligned} \quad (4.27)$$

Taking the expected value on both sides of (4.27) and applying Fubini's theorem we arrive at

$$\begin{aligned} \varphi(t) &= x + \int_0^t \big\{ -\delta\big(\varphi(s-) + \frac{B(s)}{2A(s)}\big) + r\varphi(s-) \\ &\quad + NC(s) + \kappa AL(s) - \kappa\varphi(s-)\big\} ds, \end{aligned} \quad (4.28)$$

where we define for $0 \leq s \leq T$

$$\varphi(s) = \mathbb{E}^{0,x,\lambda}[X^{\hat{\pi}}(s)], \quad (4.29)$$

$$NC(s) = e^{-\rho(T-s)} f(s) \mathbb{E}^{0,\lambda}[Da(\lambda(T))], \quad (4.30)$$

$$AL(s) = e^{-\rho(T-s)} F(s) \mathbb{E}^{0,\lambda}[Da(\lambda(T))], \quad (4.31)$$

and

$$\begin{aligned} B(s) &= -\beta e^{(r-\kappa-\delta)(T-s)} + \big(-2e^{(r-\kappa-\delta)(T-s)} \\ &\quad + 2e^{(r-\kappa-\rho)T + (r-\kappa-\delta)(T-s)} \int_s^T e^{(\kappa+\rho-r)u}\big(f(u) + \kappa F(u)\big) du\big) \mathbb{E}^{0,\lambda}[Da(\lambda(T))]. \end{aligned} \quad (4.32)$$

We point out that the expected values of the stochastic integrals in (4.27) are indeed equal to zero due to the square integrability of the process $X^{\hat{\pi}}$ (lemma 4.2) and the $\mathbb{P}$-a.s. uniform boundness of $B(s,\lambda(s))/2A(s)$, whereas (4.30)-(4.32) arise due to the Markov property and the law of iterated expectations.

It is easy to show that the function φ satisfying (4.28) must be continuous and differentiable. The integral equation (4.28) can be transformed back into the ordinary differential equation

$$\frac{d\varphi}{dt}(t) = \big(r - \delta - \kappa\big)\varphi(t) - \delta\frac{B(t)}{2A(t)} + NC(t) + \kappa AL(t), \quad \varphi(0) = x, \tag{4.33}$$

which can be solved resulting in

$$\varphi(T) = xe^{(r-\delta-\kappa)T} + \int_0^T \big(-\delta\frac{B(t)}{2A(t)} + NC(t) + \kappa AL(t)\big)e^{(r-\delta-\kappa)(T-t)}dt. \tag{4.34}$$

It is left to find the value of β such that the constraint $\varphi(T) = \mathbb{E}^{0,\lambda}[Da(\lambda(T))]$ is satisfied. With a little algebra we arrive at the value of the Lagrange multiplier

$$\begin{aligned}\beta &= \frac{2e^{(r-\kappa)T}}{e^{\delta T}-1}\left((\rho - r)e^{-\rho T}\int_0^T e^{(\rho-r+\kappa)s}F(s)ds\mathbb{E}^{0,\lambda}[Da(\lambda(T))] - x\right)\\ &= \frac{2e^{(r-\kappa)T}}{e^{\delta T}-1}\left(\frac{\rho - r}{\kappa+\rho-r}\big(e^{-(r-\kappa)T} - e^{-\rho T}\int_0^T e^{(\rho-r+\kappa)s}f(s)ds\big)\mathbb{E}^{0,\lambda}[Da(\lambda(T))] - x\right),\end{aligned} \tag{4.35}$$

where the similar identity to (4.23) is applied. Let us notice that

$$\begin{aligned}\beta &\geq \frac{2e^{(r-\kappa)T}}{e^{\delta T}-1}\left((\rho - r)e^{-\rho T}\int_0^T e^{(\rho-r)s}F(s)ds\mathbb{E}^{0,\lambda}[Da(\lambda(T))] - x\right)\\ &= \frac{2e^{-\kappa T}}{e^{\delta T}-1}\left(\mathbb{E}^{0,\lambda}[Da(\lambda(T))] - \int_0^T e^{r(T-s)}NC(s)ds - xe^{rT}\right),\end{aligned} \tag{4.36}$$

provided $\rho \geq r$. Moreover, in the case of $\kappa = 0$ we have that

$$\beta = \frac{2}{e^{\delta T}-1}\left(\mathbb{E}^{0,\lambda}[Da(\lambda(T))] - \int_0^T e^{r(T-s)}NC(s)ds - xe^{rT}\right). \tag{4.37}$$

It is reasonable to assume that the plan's valuation rate ρ would be chosen, such that the term in the bracket of (4.37) would be positive. Otherwise, the future contributions (normal costs) invested in the bank account would be sufficient, on average, to cover the liability. The rate ρ should be set to reflect the plan manager's expectations concerning higher returns in the financial market and the future inflow of supplementary contributions. In this case, the Lagrange multiplier would be positive due to (4.36).

Below we state the theorem summarizing our results.

Theorem 4.2. *The investment strategy given by*

$$\hat{\pi}(t) = -\bar{\delta}\left(X^{\hat{\pi}}(t-) + \frac{B(t,\lambda(t))}{2A(t)}\right), \quad \bar{\delta} = \frac{\mu - r}{\sigma^2 + \int_{z>-1} z^2\nu(dz)}, \tag{4.38}$$

is the optimal investment strategy for the mean-variance problem (3.3), and the minimum variance of the surplus at time T is equal to $A(0)x^2 + B(0,\lambda)x + C(0,\lambda)$. The functions A, B, C and the constant β are given by (4.16)-(4.18), (4,35), whereas $X^{\hat{\pi}}$ is the fund process under the optimal strategy, evolving according to (4.27).

Proof:
The investment strategy (4.38) is admissible, as it is a square integrable, predictable process, such that the optimal fund process $X^{\hat{\pi}}$ is unique, see lemma 4.2. Our solution of the Hamilton-Jacobi-Bellman equation, in the form of $A(t)x^2 + B(t,\lambda)x + C(t,\lambda)$, is smooth as required, see lemma 4.1. The conditions (4.6), (4.7) and (4.12) are clearly satisfied, by the method of constructing the solution, and (4.15) is indeed the minimizer of the quadratic function in π. Finiteness of the fourth moment of $X^{\hat{\pi}}$, established in lemma 4.2, guarantees the conditions (4.8), (4.9), as well as the uniform integrability of the sequence in (4.10) and (4.13). We can conclude that the strategy (4.38) is optimal for the optimization problem (4.5). The parameter β is chosen such that the constraint on the expected value of the surplus is satisfied and, as a result, the value function for (4.5) at time 0 is equal to the minimum variance of the surplus at time T. □

We end up with giving the interpretation of the investment strategy (4.38). The derived optimal strategy has the well-known form, which is typical for quadratic control problems. However, it is modify and takes into account the underlying financial market and the stochastic mortality.

Notice that the effect of the jumps on the optimal investment strategy is unclear, as all $\bar{\delta}, A$ and B involve the measure ν. In Ngwira, Gerrard (2006) the effect of the mean jump magnitude on the asset allocation is investigated numerically. They conclude that as the mean jump magnitude becomes larger in absolute value, the optimal allocation in the risky asset decreases. The pension manager who applied the investment strategy, which is optimal for the lognormally distributed returns, in the market in which jumps can occur, would take too much financial risk. Moreover, the optimal allocation is highly sensitive to changes in the mean jump magnitude if the mean jump magnitude takes very small positive or negative values. The manager should be concerned about small discontinuities in the price quotations, which are always observable, and should apply the appropriate strategy which will protect the fund from the loss in the case of large drawdown movement in the stock.

The effect of the mortality intensity on the optimal strategy can easily be studied. One should only consider mortality processes under which the expected liability $Da(t,\lambda)$ is an decreasing function of λ, as the lower mortality must yield the longer life-time and the longer duration of the annuity payments. If this is the case, then the lower level of the intensity results in the higher amount of the fund invested in the risky asset (provided that $\rho \geq r$). Moreover, the lower level of the intensity results in the higher normal cost (2.11) and the higher supplementary cost (2.13) as well. If the future life-time of the pensioners increases, the manager should collect higher contributions and take greater financial risk in order to accumulate sufficient funds to cover the claim. The pension manager who does not follow this adjustments is very likely not to fullfill its commitments.

Finally, it is straightforward to note that the constraint concerning the expected value of the surplus leads to the higher allocation of the fund in the risky asset, provided that $\beta > 0$.

Example 4.1. We continue example 2.1. We assume that the stock price follows an exponential Variance Gamma process of the form

$$S(t) = e^{0,28t+L(t)}, \quad L(t) = -0,2h(t) + 0,2W(h(t)), \tag{4.39}$$

where $h(t)$ is a Gamma distributed random variable with the density function

$$g_{h(t)}(y) = \frac{1}{\Gamma(t/0,003)(0,003)^{t/0,003}} y^{\frac{t}{0,003}-1} e^{-\frac{y}{0,003}}. \tag{4.40}$$

For the subordinated Brownian motion representation of a Variance Gamma process we refer the reader to chapter 2.3 in Kyprianou *et al* (2005). This choice of parameters corresponds to $\mu = 0,1$.

In table 2, based on our simulation results, we give some quantities of the empirical distribution of the ratio $X(T)/Da(\lambda(T))$ in two cases: a) when the fund is controlled in order to minimize the mean square hedging error (3.1) and the stochastic nature of mortality is not taken into account in the derivation of the optimal strategies, $\beta = 0, \mathbb{E}^{t,\lambda}[Da(\lambda(T))] = \mathbb{E}^{0,\lambda}[Da(\lambda(T))] = 11,901$; b) when the fund is controlled according to the strategy (4.38) in order to minimize the variance of the surplus and when the stochastic nature of mortality is taken into account as well. We assume that $\kappa = 0, \rho = 0,08, D = 1000, x_0 = 500$ and $f(t) = 1/20$.

Table 2: Distribution of the ratio $X(T)/a(\lambda(T))$

	Case "a"	Case "b"
Mean value	95,096%	100,926%
Standard deviation	6,508%	10,477%
1st percentile	70,476%	60,344%
5th percentile	86,571%	86,682%
10th percentile	91,400%	94,549%
90th percentile	99,726%	107,608%
95th percentile	100,252%	108,019%
99th percentile	101,029%	108,824%

First of all one should notice that in the case "a" the accumulated fund is not sufficient, on average, to cover the liability. The average deficit in the terminal surplus is about -566,5. However, due to the additional constraint, the variance of the ratio in case "b" is higher. The terminal constraint and positive value of β lead to the strategy of investing higher amounts in the stock and this explains the increase in the mean value and standard deviation for the case "b", compared with the case "a". The Lagrange multiplier in our example is equal to $\beta = 1975$. It should also be noticed that the distribution of the ratio in the case "b" has a thicker left tail (1st percentile), as well as the right tail (90th, 95th, 99th percentiles), compared with the case "a". □

5 Generalized mean-variance optimization problem

In this section we solve the generalized problem (3.4) and find the optimal investment and contribution strategy which minimize the variance of the surplus along with the expected value of squares of future contribution rates. The problem (3.4) can be transformed, as in section 4, by using a Lagrange multiplier. First we solve the stochastic control problem

$$\begin{aligned}\inf_{(\pi,u)\in\mathcal{B}} \mathbb{E}^{0,x,\lambda}\Big[\textstyle\int_0^T u^2(t)dt + \alpha\big(X^{\pi,u}(T) - Da(\lambda(T))\big)^2 \\ -\gamma\big(X^{\pi,u}(T) - Da(\lambda(T))\big)\Big], \qquad (5.1)\end{aligned}$$

find the optimal strategy $(\tilde{\pi}, \tilde{u})$ and then choose a Lagrange multiplier γ such that

$$\mathbb{E}^{0,x,\lambda}\big[X^{\tilde{\pi},\tilde{u},\gamma}(T) - Da(\lambda(T))\big] = 0. \qquad (5.2)$$

Mathematical details of arriving at the solution of (5.1) and (5.2) are very similar to those in section 4 and are omitted. They can be obtained from the authors upon request.

The differential operator $\mathcal{L}_M$ remains the same, whereas the integro-differential operator $\mathcal{L}_F$ takes a slightly different form

$$\begin{aligned}\mathcal{L}_F^{\pi,u}\phi(t,x,\lambda) &= \big(\pi(\mu - r) + xr + NC(t,\lambda) + u\big)\frac{\partial\phi}{\partial x}(t,x) \\ &+ \frac{1}{2}\pi^2\sigma^2\frac{\partial^2\phi}{\partial x^2}(t,x) \\ &+ \int_{z>-1}\big(\phi(t,x+\pi z) - \phi(t,x) - \pi z\frac{\partial\phi}{\partial x}(t,x)\big)\nu(dz). \qquad (5.3)\end{aligned}$$

We define the new set of admissible strategies.

Definition 5.1. *A strategy $(\pi(t), u(t), 0 < t \le T)$ is admissible, $(\pi, u) \in \mathcal{B}$, if it satisfies the following assumptions:*

1. $\pi : (0,T] \times \Omega \to \mathbb{R}$ *and* $u : (0,T] \times \Omega \to \mathbb{R}$ *are predictable mappings with respect to filtration* $\mathbb{F}$,

2. $\mathbb{E}^{0,x,\lambda}[\int_0^T \pi^2(t)dt] + \mathbb{E}^{0,x,\lambda}[\int_0^T u^2(t)dt] < \infty$,

3. *the stochastic differential equation (2.6) has a unique solution* $X^{\pi,u}$ *on* $[0,T]$.

The verification theorem 4.1 can be extended to include the running cost of u. It is not difficult to realize that the Hamilton-Jacobi-Bellman equation for the optimization problem (5.1) is of the form

$$0 = \min_{(\pi,u)\in\mathbb{R}^2} \big\{u^2 + \frac{\partial w}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\pi,u} w(t,x,\lambda) + \mathcal{L}_M w(t,x,\lambda)\big\}. \tag{5.4}$$

As in the previous section we try to find a quadratic solution $w(t,x,\lambda) = P(t,\lambda)x^2 + Q(t,\lambda)x + R(t,\lambda)$. The optimal strategies for which the minimum on the right hand side of (5.4) is attained are given by

$$\tilde{u}(t,x,\lambda) = -\frac{Q(t,\lambda)}{2} - P(t,\lambda)x, \tag{5.5}$$

$$\tilde{\pi}(t,x,\lambda) = -\bar{\delta}\big(x + \frac{Q(t,\lambda)}{2P(t,\lambda)}\big). \tag{5.6}$$

Substituting (5.5), (5.6) into (5.4) and collecting the terms we arrive at the following partial differential equations

$$\begin{cases} 0 = \frac{\partial P}{\partial t}(t,\lambda) + \mathcal{L}_M P(t,\lambda) + (2r-\delta)P(t,\lambda) - P^2(t,\lambda), \\ P(T,\lambda) = \alpha, \end{cases} \tag{5.7}$$

$$\begin{cases} 0 = \frac{\partial Q}{\partial t}(t,\lambda) + \mathcal{L}_M Q(t,\lambda) + \big(r - \delta - P(t,\lambda)\big)Q(t,\lambda) + 2P(t,\lambda)NC(t,\lambda), \\ Q(T,\lambda) = -\gamma - 2\alpha Da(\lambda), \end{cases} \tag{5.8}$$

$$\begin{cases} 0 = \frac{\partial R}{\partial t}(t,\lambda) + \mathcal{L}_M R(t,\lambda) - \frac{Q^2(t,\lambda)}{4P(t,\lambda)}(P(t,\lambda) + \delta) + Q(t,\lambda)NC(t,\lambda), \\ R(T,\lambda) = \alpha D^2 a^2(\lambda) + \gamma Da(\lambda). \end{cases} \tag{5.9}$$

We can solve the equation (5.7) explicitly and arrive at

$$P(t) = \frac{1}{\frac{1}{\alpha}e^{(\delta-2r)(T-t)} + \frac{1}{\delta-2r}\big(e^{(\delta-2r)(T-t)} - 1\big)}. \tag{5.10}$$

We can also conclude that the partial differential equations (5.8) and (5.9) have unique solutions in the class $\mathcal{C}$. We state the Feynman-Kac representation of the unique solution to (5.8)

$$\begin{aligned} Q(t,\lambda) &= -\gamma e^{(r-\delta)(T-t)-\int_t^T P(s)ds} + \big(- 2\alpha e^{(r-\delta)(T-t)-\int_t^T P(s)ds} + 2e^{-\rho(T-t)} \\ &\quad \times \int_t^T P(s)f(s)e^{(r+\rho-\delta)(s-t)-\int_t^s P(u)du}ds\big)\mathbb{E}^{t,\lambda}[Da(\lambda(T))]. \end{aligned} \tag{5.11}$$

It is easy to check that the denominator in (5.10) is positive and bounded away from zero. It is also not difficult to realize that $-\log(\frac{1}{\alpha}+\frac{1}{\delta-2r}-\frac{1}{\delta-2r}e^{(2r-\delta)(T-t)})$ is the antiderivative of $P(t)$ and that the following equality holds

$$\int_t^s P(u)du = \log\frac{P(s)}{P(t)} + (2r-\delta)(s-t). \tag{5.12}$$

Applying (5.12) in (5.11) we can arrive at

$$Q(t,\lambda) = -P(t)e^{-r(T-t)}\Big\{\frac{\gamma}{\alpha} + 2\big(1-e^{(r-\rho)T}\int_t^T e^{(\rho-r)s}f(s)ds\big)\mathbb{E}^{t,\lambda}[Da(\lambda(T))]\Big\}. \tag{5.13}$$

Notice that Q is non-positive in the case of $\rho \geq r$ and $\gamma \geq 0$, as the following trivial equality holds

$$e^{(r-\rho)T}\int_t^T e^{(\rho-r)s}f(s)ds \leq 1, \quad t\in[0,T]. \tag{5.14}$$

It is left to find a Lagrange multiplier γ. The dynamics of the fund $X^{\tilde{\pi},\tilde{u}}$ under the optimal strategy $(\tilde{\pi},\tilde{u})$ are given by

$$\begin{aligned} dX^{\tilde{\pi},\tilde{u}}(t) &= \Big\{-\delta\big(X^{\tilde{\pi},\tilde{u}}(t-)+\frac{Q(t,\lambda(t))}{2P(t)}\big) + X^{\tilde{\pi},\tilde{u}}(t-)r + NC(t,\lambda(t)) \\ &\quad -P(t)X^{\tilde{\pi},\tilde{u}}(t-) - \frac{1}{2}Q(t,\lambda(t))\Big\}dt \\ &\quad -\bar{\delta}\big(X^{\tilde{\pi},\tilde{u}}(t-)+\frac{Q(t,\lambda(t))}{2P(t)}\big)\big(\sigma dW(t) + \int_{z>-1} z\tilde{M}(ds\times dz)\big) \end{aligned} \tag{5.15}$$

with the initial condition $X(0)=x$. As in section 4 we can derive the ordinary differential equation for $\psi(t)=\mathbb{E}^{0,x,\lambda}[X^{\tilde{\pi},\tilde{u}}(t)]$, which is

$$\frac{d\psi}{dt}(t) = \big(r-\delta-P(t)\big)\psi(t) - \big(\frac{\delta}{P(t)}+1\big)\frac{Q(t)}{2} + NC(t), \quad \psi(0)=x, \tag{5.16}$$

where

$$Q(t) = -P(t)e^{-r(T-t)}\Big\{\frac{\gamma}{\alpha} + 2\big(1-e^{(r-\rho)T}\int_t^T e^{(\rho-r)s}f(s)ds\big)\mathbb{E}^{0,\lambda}[Da(\lambda(T))]\Big\}. \tag{5.17}$$

With algebraic manipulations, we can arrive at the value of the Lagrange multiplier

$$\gamma = 2\alpha\frac{\big(\alpha e^{2rT}+(\gamma_3-\gamma_2)e^{(r-\rho)T}-\gamma_1\big)\mathbb{E}^{0,\lambda}[Da(\lambda(T))] - xe^{rT}P(0)}{\gamma_1}, \tag{5.18}$$

where

$$\gamma_1 = \int_0^T e^{2rt}(\delta+P(t))P(t)dt, \tag{5.19}$$

$$\gamma_2 = \int_0^T e^{(\rho+r)t}P(t)f(t)dt, \tag{5.20}$$

$$\gamma_3 = \int_0^T e^{2rt}(\delta+P(t))P(t)\int_t^T e^{(\rho-r)s}f(s)dsdt. \tag{5.21}$$

We conclude with the following theorem.

Theorem 5.1. *The investment strategy*

$$\tilde{\pi}(t) = -\bar{\delta}\left(X^{\tilde{\pi},\tilde{u}}(t-) + \frac{Q(t,\lambda(t))}{2P(t)}\right), \quad \bar{\delta} = \frac{\mu - r}{\sigma^2 + \int_{z>-1} z^2 \nu(dz)}, \tag{5.22}$$

and the supplementary contribution rate

$$\tilde{u}(t) = -\frac{Q(t,\lambda(t))}{2} - P(t)X^{\tilde{\pi},\tilde{u}}(t-), \tag{5.23}$$

is the optimal strategy for the generalized mean-variance problem (3.4), and the minimum cost is equal to $P(0)x^2 + Q(0,\lambda)x + R(0,\lambda)$*. The functions* P*,* Q*,* R *and the constant* γ *are given by (5.7)-(5.9), (5.18), whereas* $X^{\tilde{\pi},\hat{u}}$ *is the fund process under the optimal strategy, evolving according to (5.15).*

The effect of the terminal constraint, the jumps and the intensity on the optimal investment strategy is the same as in section 4. Notice that, as expected, the optimal supplementary costs (5.23) is the decreasing function of λ; the lower mortality intensity yields the higher contribution (provided that $\rho \geq r$).

6 Conclusions

In this paper we have investigated a retirement plan of a defined benefit type which accumulated funds are converted into annuities for participants. We have considered the price of the annuity as a random variable which randomness arises due to stochastic evolution of a mortality intensity. We have assumed that the asset return is driven by a Lévy process. We believe that this are very important extensions as far as pension modelling is concerned. We have solved two new optimization problems and arrived at the optimal strategies and the optimal value functions.

We have only dealt with one cohort of participants and it might be desirable to extend the results and incorporate more cohorts of workers who continuously join the plan. This means modelling mortality intensities for each age and would lead to the multidimensional (or even infinitely dimensional) process of the mortality intensity. Deriving a result in this framework seems to be challenging. This is left for further research.

Even though we have presented solutions for one cohort, we believe that they might be useful in the management of defined benefit pension plans. The simple heuristic strategy for an aggregate pension plan is to apply our results, at each point of time, for each of the cohort separately, and then rebalance the individual accounts with any profits or losses which arises when paying out a benefit for retiring members.

Finally, we would like to refer the interested reader to the paper of Delong (2007) where indifference pricing of a life insurance portfolio is investigated, in the case when mortality follows a diffusion process and an insurer invests in a financial market with an asset which price is driven by a Lévy process.

References

[1] Applebaum, D., 2004. *Lévy Processes and Stochastic Calculus*. Cambridge University Press.

[2] Ballotta, L., Haberman, S., 2006. The fair valuation problem of guaranteed annuity options: the stochastic mortality environment case. *Insurance: Mathematics and Economics* 38, 195-214.

[3] Becherer, D., Schweizer, M., 2005. Classical solutions to reaction-diffusion systems for hedging problems with interacting Itô processes. *The Annals of Applied Probability* 15, 1111-1144.

[4] Bielecki, T., Jeanblanc, M., Rutkowski, M., 2004. Mean-variance hedging of defaultable claims, preprint.

[5] Bielecki, T., Jin, H., Pliska, S., Zhou, X.Y., 2005. Dynamic mean-variance with portfolio selection with bankruptcy prohibition. *Mathematical Finance* 15, 213-244.

[6] Cairns, A., 2000. Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. *ASTIN Bulletin* 30, 19-55.

[7] Cairns, A., Blake, D., Dowd, K., 2004a. Stochastic lifestyling: Optimal asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control* 30, 843-877.

[8] Cairns, A., Blake, D., Downd, K., 2004b. Pricing frameworks for securitization of mortality risk. preprint.

[9] Chang, S.C., Tzeng, L.T., Miao, J.C.Y., 2003. Pension funds incorporating downside risk. *Insurance: Mathematics and Economics* 32, 217-228.

[10] Cont, R., Tankov, P., 2004. *Financial Modelling with Jump Processes*. Chapman & Hall.

[11] Dahl, M., 2004. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. *Insurance: Mathematics and Economics* 35, 113-136.

[12] Delong, Ł, 2007. Indifference pricing of a life insurance portfolio with systematic mortality risk in a market with an asset driven by a Lévy noise. revised form submitted to *Scandinavian Actuarial Journal*.

[13] Delong, Ł., Gerrard, R., 2007. Mean-variance portfolio selection for a non-life insurer. *Mathematical Methods of Operations Research*. in press.

[14] Duffie, D., Richardson, H., 1991. Mean-variance hedging in continuous time. *The Annals of Applied Probability* 1, 1-15.

[15] Föllmer, H., Sonderman, D., 1986. Hedging of non-redundant contingent claims. In Mas-Colell, A., Hildebrand, W.. *Contributions to Mathematical Economics*. Amsterdam: North Holland.

[16] Ngwira, B., Gerrard, R., 2006. Stochastic pension fund control in the presence of Poisson jumps. *Insurance: Mathematics and Economics*. in press.

[17] Haberman, S., Butt, Z., Megaloudi, C., 2000. Contribution and solvency risk in a defined benefit pension scheme. *Insurance: Mathematics and Economics* 27, 237-259.

[18] Haberman, S., Sung, J.H., 1994. Dynamic approaches to pension funding. *Insurance: Mathematics and Economics* 15, 151-162.

[19] Haberman, S., Sung, J.H., 2005. Optimal pension funding dynamics over infinite control horizon when stochastic rates of return are stationary.*Insurance: Mathematics and Economics* 36, 103-116.

[20] Heath, D., Schweizer, M., 2000. Martingales versus PDE's in finance: an equivalent result with examples. *Journal of Applied Probability* 37, 947-957.

[21] Hipp, C., Taksar, M., 2005. Hedging in incomplete markets and optimal control. preprint.

[22] Josa-Fombellida, R., Rincón-Zapatero, J., 2004. Optimal risk management in defined benefit stochastic pension funds. *Insurance: Mathematics and Economics* 34, 489-503.

[23] Kohlmann, M., Peisl, B., 1999. A note on mean-variance hedging of non-attainable claims. preprint.

[24] Kyprianou, A., Schoutens, W., Wilmott, P., 2005. *Exotic Option Pricing and Advanced Lévy Models*. Wiley.

[25] Lim, A., 2004. Quadratic hedging and mean-variance portfolio selection in an incomplete market. *Mathematics of Operations Research* 29, 132-161.

[26] Luciano, E., Vigna, E., 2005. Non mean reverting processes for stochastic mortality. preprint.

[27] Markowitz, H., 1952. Portfolio selection. *Journal of Finance* 7, 77-91.

[28] Øksendal, B., Sulem, A., 2005. *Applied Stochastic Control of Jump Diffusions*. Springer.

[29] Protter, P., 2005. *Stochastic Integration and Differential Equations*. Springer.

[30] Schrager, D., 2006. Affine stochastic mortality. *Insurance: Mathematics and Economics* 38, 81-97.

[31] Schweizer, M., 1992. Mean-variance hedging of general claims. *The Annals of Applied Probability* 1, 171-179.

[32] Schweizer, M., 1996. Approximation pricing and the variance optimal martingale measure. *The Annals of Probability* 64, 206-236.

[33] Zhou, X.Y., Li, D., 2000. Continuous time mean- variance portfolio selection: a stochastic LQ framework. *Applied Mathematics Optimization* 42, 19-33.