

Optimal Investment for a Defined-Contribution Pension Scheme under a Regime Switching Model

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Abstract

We study an asset allocation problem for a defined-contribution (DC) pension scheme in its accumulation phase. We assume that the amount contributed to the pension fund by a pension plan member is coupled with the salary income which fluctuates randomly over time and contains both a hedgeable and non-hedgeable risk component. We consider an economy in which macroeconomic risks are existent. We assume that the economy can be in one of I states (regimes) and switches randomly between those states. The state of the economy affects the dynamics of the tradeable risky asset and the contribution process (the salary income of a pension plan member). To model the switching behavior of the economy we use a counting process with stochastic intensities. We find the investment strategy which maximizes the expected exponential utility of the discounted excess wealth over a target payment, e.g. a target lifetime annuity.

Keywords: Exponential utility maximization, macroeconomic risks, certainty equivalent, Backward Stochastic Differential Equations.

1 Introduction

The ongoing shift from defined benefit to defined contribution (DC) plans in many developed countries has pushed the optimal asset allocation problem for a DC plan to the front of risk management of occupational retirement plans. It is important to recognize that a large part of the risk borne by a pension plan is systematic and depends on economic cycles. It is empirically observed that the deficit of pension benefits grows during an economic downturn, while it is more likely to have a surplus in an economic boom. In other words, the financial position of a pension plan is strongly subject to macroeconomic risks.¹ It is clear that the variables like the mean and the volatility of asset returns (and consequently the mean and the volatility of pension funds) vary substantially between diverse economic states. There are mixed empirical observations about how the volatility changes with economic situations, yet many researchers have shown the countercyclical behavior of the stock volatility, see e.g. Schwert (1989) and Engle et. al. (2008). Moreover, macroeconomic variables like employment, inflation and interest rate, which are strongly related with the economic situation, have substantial effect on the mean of stock returns, see e.g. Campbell (1987) and Asprem (1989).

Actuaries have developed ways to manage the risks related to economic and demographic indices in the long term, but to the best of our knowledge, macroeconomic risks have not been modelled explicitly in the actuarial literature on optimal asset allocation for DC plans. In this paper we incorporate macroeconomic risks into an asset allocation problem for a DC pension scheme by considering a multi-state regime-switching financial model. We investigate an economy which can be in one of I states (regimes) and switches randomly between those states. The state of the economy affects the dynamics of the tradeable risky asset and the contribution process (and the salary income of a pension plan member). To model the switching behavior for the states of the economy, we use a

¹Unlike pension plans, the theory of bonus in life insurance refers to an entirely different paradigm: policyholders get only what the realized interest and mortality over the tract period can sustain (see e.g. Norberg (2001)). Hence, there is no environment risk on the part of the company.

counting process with stochastic intensities. The transition intensities depend not only on the current state of the economy but also on the current price of the risky asset. Hence, we model an effect in which not only the stock price is affected by the transitions between the states of the economy but also the stock price determines the transition intensities, see Elliott et. al. (2011) for a financial motivation of a so-called feedback effect. The asset allocation problem for a DC plan differs from the standard asset allocation problem for an investor (see e.g. Merton (1969), Merton (1971)) since we have to consider an additional random stream of contributions which flows to the pension fund. The amount contributed to the pension fund by a pension plan member is modelled by a process which contains a tradeable risk component and non-tradeable risk components. The tradeable risk component can be hedged with the tradeable risky asset, and the non-tradeable risk components represent unhedgeable continuous fluctuations in the contribution rate and the unhedgeable switching behavior of the economy. The pension fund's objective is to maximize the expected exponential utility of the discounted excess wealth over a target payment at the retirement age.² The target payment can be chosen as a lifetime annuity with the benefit based on the the final salary of the pension beneficiary. Let us remark that maximizing the exponential utility of the excess wealth is related to minimizing the probability that the terminal wealth falls below the target level.³

In this paper we use Backward Stochastic Differential Equations (BSDEs) to solve the optimization problem. To solve our exponential utility maximization problem we follow Hu et. al. (2005) and Becherer (2006) and adapt their techniques to our setting. We characterize the optimal investment strategy for a DC pension scheme and the optimal value function for the optimization problem with a solution to a non-linear backward stochastic differential equation with a Lipschitz generator. From the mathematical point

²In this paper, we assume there is no agency problem, i.e. the pension fund (manager) and the pension beneficiary share the same optimization objective.

³As pointed out e.g. by Browne (1995), maximizing exponential utility of the wealth at a given terminal time is intrinsically related to maximizing the survival probability, which is equivalent to minimizing the probability of ruin. If we maximize the excess wealth (terminal wealth minus a target payment), this is then related to minimizing the probability that the terminal wealth falls below the target level.

of view the novelty of the paper is that we derive the solution to the optimization problem in a new model and we investigate properties of the solution to the BSDE which arises in our calculations. We would like to point out that our model is different from Hu et. al. (2005) and Becherer (2006). We consider a regime-switching economy, i.e. a risky asset price dynamics driven with a Brownian motion with coefficients depending on a counting process, and we allow for a stochastic flow of contributions. Hu et. al. (2005) consider a risky asset price dynamics based on a Brownian motion, and Becherer (2006) considers a dynamics based on an abstract random measure, and the authors do not consider a contribution process. Our financial model is set up with the goal that the results derived in it can be beneficial for a manager of a DC pension scheme. We also would like to point out that the solution from Section 11.1 from Delong (2013) cannot be applied in our setting due to differences in the models, e.g. in Delong (2013) the coefficients of the risky asset price dynamics do not depend on the jump process which is used to model an insurance risk.

Although we are not able to model the economy very realistically in a long time perspective, the proposed framework and the solution developed can still give pension plan managers some insights and guidance at a general, qualitative level. We aim to model a plausible optimization problem of a DC pension scheme beneficiary in a fairly general setting which captures prevalent beliefs about the workings of the market (in macro). We do not recommend pension funds to adopt the suggested optimal investment strategy quantitatively.

There already exists a stream of literature on optimal asset allocation for pension funds. For instance, Gao (2008) studies an asset allocation problem under a stochastic interest rate. Boulier et. al. (2001) incorporate a constraint into an investment problem under which a guaranteed benefit is provided to a pension beneficiary. Blake et. al. (2012) investigate an asset allocation problem under a loss-averse preference. Cairns et. al. (2006) consider a stochastic salary income of a pension beneficiary and find the investment

strategy which maximizes the expected power utility of the ratio of the terminal fund and the terminal salary. The closest to our research is the paper by Korn et. al. (2011), who investigate a utility optimization problem for a DC pension plan with a stochastic salary income and a stochastic contribution process in a regime-switching economy. Their main interest lies in solving a filtering problem since they assume that the states of the economy are modelled by a hidden Markov chain. We would like to point out that Korn et. al. (2011) assume constant volatilities of the asset returns and the contribution process and constant intensities of the Markov chain. They provide an explicit investment strategy for a logarithmic utility. In this paper we consider more general dynamics of the tradeable asset, the contribution process and the Markov chain with volatilities and intensities depending on the (observable) states of the economy, and we derive the optimal investment strategy for an exponential utility. Let us recall that from the macroeconomic point of view it is very important to assume countercyclical behavior of the volatility of the stock and the dependence of the transition intensity on the stock price, see Schwert (1989), Engle et. al. (2008), Elliott et. al. (2011). We would like to point out that since we use BSDEs to solve our asset allocation problem the results of this paper can be easily extended for a model with more general - even non-Markovian - dynamics.

The paper is organized as follows. Section 2 describes the financial market under the macroeconomic risks and formulates the asset allocation problem. In Section 3, we solve the investment problem and we find the optimal investment strategy which maximizes the expected exponential utility of the discounted excess wealth over a target payment. In Section 4, we present some numerical results. Finally, Section 5 provides some concluding remarks.

2 The model

We deal with a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and a finite time horizon $T < \infty$. We assume that \mathbb{F} satisfies the usual hypotheses of completeness

(\mathcal{F}_0 contains all sets of \mathbb{P} -measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define an \mathbb{F} -adapted, two-dimensional standard Brownian motion $(W_1, W_2) = (W_1(t), W_2(t), 0 \leq t \leq T)$ and an \mathbb{F} -adapted, multivariate counting process $N = (N_1(t), \dots, N_I(t), 0 \leq t \leq T)$. The one-dimensional Brownian motions W_1 and W_2 are independent. The one-dimensional counting processes (N_1, \dots, N_I) are not independent, and the dependence structure is described in the sequel.

We consider an economy which can be in one of I states (regimes) and switches randomly between those states. For $i = 1, \dots, I$, the counting process N_i counts the number of transitions of the economy into state i . Furthermore, let $J = (J(t), 0 \leq t \leq T)$ denote an \mathbb{F} -adapted process which indicates the current state of the economy. If the economy is in regime $k \in \{1, \dots, I\}$ at the initial point of time, then the dynamics of the process J is given by the stochastic differential equation

$$dJ(t) = \sum_{i=1}^I (i - J(t-)) dN_i(t), \quad J(0) = k \in \{1, \dots, I\}.$$

A pension plan manager manages a pension fund and trades in a financial market. The financial market consists of a risk-free bank account and a risky asset. In the sequel we only consider discounted quantities. Hence, the discounted value of the bank account is constant. We assume that the dynamics of the discounted value of the risky asset $S = (S(t), 0 \leq t \leq T)$ is given by the stochastic differential equation

$$dS(t) = \mu(J(t-))S(t)dt + \sigma(J(t-))S(t)dW_1(t), \quad S(0) = s, \quad (2.1)$$

where

(A1) $(\mu(i))_{i=1, \dots, I}$ is a sequence of real numbers and $(\sigma(i))_{i=1, \dots, I}$ is a sequence of strictly positive numbers,

which describe the value of the drift μ and the volatility σ of the discounted risky asset (2.1) if the economy is in regime i . The drift and the volatility of the risky asset depend on

the state of the economy as it is observed in empirical data and justified by macroeconomic theory, see the Introduction.

We comment on the dynamics (2.1). The dynamics for the discounted value of the risky asset can be motivated in the following way. Let us assume that the value of the bank account $B = (B(t), 0 \leq t \leq T)$ is modelled by the equation

$$dB(t) = r(J(t-))B(t)dt,$$

where the interest rate r is stochastic and depends on the state of the economy. Let the value of the risky asset $S^{undisc} = (S^{undisc}(t), 0 \leq t \leq T)$ satisfy the stochastic differential equation

$$dS^{undisc}(t) = \tilde{\mu}(J(t-))S^{undisc}(t)dt + \sigma(J(t-))S^{undisc}(t)dW(t),$$

where $\tilde{\mu}$ is the real-world drift of the asset. Then, the discounted value of the risky asset $S(t) = e^{-\int_0^t r(J(s))ds} S^{undisc}(t)$ follows the dynamics

$$\begin{aligned} dS(t) &= (\tilde{\mu}(J(t-)) - r(J(t-)))S(t)dt + \sigma(J(t-))S(t)dW(t) \\ &= \mu(J(t-))S(t)dt + \sigma(J(t-))S(t)dW(t), \end{aligned}$$

which agrees with (2.1).

We now characterize the intensities of the counting process. We assume that

(A2) for $i = 1, \dots, I$, the counting process N_i has intensity $\lambda_i(J(t-), S(t))$ where $\lambda_i : \{1, \dots, i-1, i+1, \dots, I\} \times [0, \infty) \mapsto \mathbb{R}$ is a bounded mapping.

Consequently, the compensated counting processes

$$\tilde{N}_i(t) = N_i(t) - \int_0^t \lambda_i(J(s-), S(s))ds, \quad 0 \leq t \leq T, \quad i = 1, \dots, I,$$

are \mathbb{F} -martingales. We remark that $\lambda_i(j, s)$ denotes an intensity of the transition of the economy into state i if the economy is in state j and the discounted value of the risky asset is s . The dependence of the transition intensity on the current state of the economy is obvious. The dependence of the transition intensity on the risky asset is more sophisticated. It models a so-called feedback effect in the market under which not only the risky asset (the market index) is affected by the transitions between the states of the economy but also the risky asset (the market index) determines the transition intensities, see the Introduction.

In the sequel we use the short notation:

$$\mu(t) := \mu(J(t-)), \quad \sigma(t) := \sigma(J(t-)), \quad \lambda_i(t) := \lambda_i(J(t-), S(t)), \quad 0 \leq t \leq T.$$

Let T denote the time to retirement for a pension plan member. Over the working lifetime, the next T years, the pension beneficiary receives a salary income and part of this income is contributed into the pension fund. We assume that the salary income fluctuates randomly in time. Let $G := (G(t), 0 \leq t \leq T)$ denote the discounted value of the salary income. One possibility is to assume that the dynamics of the discounted value of the salary income G is given by the stochastic differential equation

$$dG(t) = \mu_G(J(t-))G(t)dt + \sigma_G(J(t-))G(t)(\rho dW_1(t) + \sqrt{1 - \rho^2}dW_2(t)), \quad G(0) = g \quad (2.2)$$

where $(\mu_G(i))_{i=1, \dots, I}$ is a sequence of real numbers and $(\sigma_G(i))_{i=1, \dots, I}$ is a sequence of strictly positive numbers which describe the value of the drift μ_G and the volatility σ_G of the discounted salary (2.2) if the economy is in regime i , and $\rho \in [-1, 1]$ is a correlation coefficient which introduces a correlation between the returns of the salary income G and the tradeable risky asset S . We should use the second Brownian motion W_2 to model the dynamics of the salary income to take into account the fact that in real-world the returns on salary incomes and tradeable assets are not perfectly correlated

and salary incomes cannot be perfectly replicated with tradeable assets (even in a one-state economy). We do not specify the dynamics of the salary income since we are more interested in a contribution process. Let $c := (c(t), 0 \leq t \leq T)$ denote a contribution process, i.e. the discounted amount which is contributed by the pension plan member into the pension fund and is invested for his/her retirement. In a defined contribution pension plan, the contribution process c is linked to the discounted value of the salary income G . Usually the discounted contribution payment $c(t)$ into the fund is a constant proportion of the discounted salary income $G(t)$, i.e.

$$c(t) = \gamma G(t), \quad 0 \leq t \leq T. \quad (2.3)$$

However, we do not have to specify the dynamics of the contribution process. We only assume that

(A3) $c := (c(t), 0 \leq t \leq T)$ is an \mathbb{F} -adapted, positive, bounded process.

It is worth noticing that if c^{undisc} denotes an undiscounted contribution process and the interest rate r depends on the state of the economy, then we can lose the original Markovian structure if we deal with the discounted contribution $c(t) = e^{-\int_0^t r(J(s))ds} c^{undisc}(t)$.⁴ The assumption that c is \mathbb{F} -adapted means that the random amount contributed by the pension beneficiary to the pension fund contains a risk component W_1 , which can be hedged with the risky asset S , and risk components (W_2, N) , which model unhedgeable continuous fluctuations in the contribution rate and the unhedgeable switching behavior of the economy. We point out that the assumption that c be bounded can be relaxed. We introduce this assumption since it simplifies the verification of the optimality of the solution and, at the same time, it is not restrictive from a practical point of view.

Let $\pi = (\pi(t), 0 \leq t \leq T)$ denote the discounted amount of money which is invested

⁴For a moment let us assume that the boundedness assumption for c is not in force. If c^{undisc} is modelled by a CIR-like process with coefficients depending on the state of the economy, then (c, J, S) is no longer a Markov process even though (c^{undisc}, J, S) is a Markov process, the quadruple $(c, \int_0^t r(J(s))ds, J, S)$ is now a Markov process.

by the pension plan manager in the risky asset S . We call π an investment strategy of the pension fund. We know that we abuse the concept of “the strategy” from the investment point of view since in our setting π does not denote the number of units, neither the fraction of the wealth, held in the stock. However, the number of units or the fraction of the wealth which should be held in the stock can be calculated from the discounted amount of money π (if all other financial quantities are known). We introduce the set of admissible investment strategies.

Definition 2.1. *A strategy $\pi := (\pi(t), 0 \leq t \leq T)$ is called admissible, written $\pi \in \mathcal{A}$, if it satisfies the conditions:*

1. $\pi : [0, T] \times \Omega \rightarrow \mathbb{R}$ is an \mathbb{F} -predictable process,
2. $K_1(t) \leq \pi(t) \leq K_2(t)$, $0 \leq t \leq T$, where $(K_1(t), K_2(t), 0 \leq t \leq T)$ are bounded, \mathbb{F} -predictable processes.

Pension plan managers usually face constraints imposed on investment strategies. Received wisdom says that the amount invested in the risky asset should decrease over time as the pension beneficiary approaches the retirement age. Short-selling of assets is usually prohibited under the law. It is also reasonable to assume that the limits set by the manager should depend on the current state of the economy, e.g. the pension plan manager is willing to invest more in the risky asset if the economy is booming and switches to risk-free assets if the economy is busting. The set \mathcal{A} defines investment constraints which the pension plan manager has to follow in the accumulation period of the pension plan. Since we consider the discounted amount π , the bounds in \mathcal{A} are defined as general \mathbb{F} -predictable processes, see the comment and the footnote after assumption (A3). Notice that if the pension plan manager sets limits for the amount of money invested in the risky asset, then the discounted amount π is bounded. Hence, assumption 2 from Definition 2.1 is reasonable. If needed, the upper bound can be chosen to be arbitrary large and may not play a significant role in the investment decision.

We can now define the dynamics of the pension fund in the accumulation period. The discounted value of the wealth of the pension plan member $X^\pi := (X^\pi(t), 0 \leq t \leq T)$ under an admissible investment strategy $\pi \in \mathcal{A}$ is described with the stochastic differential equation

$$dX^\pi(t) = \pi(t)(\mu(t)dt + \sigma(t)dW_1(t)) + c(t)dt, \quad X^\pi(0) = x, \quad (2.4)$$

where x denotes an initial capital invested in the pension fund. We would like to comment on the dynamics (2.4). Let $(\pi^{undisc}, S^{undisc}, c^{undisc}, X^{undisc, \pi^{undisc}})$ denote the amount of money invested in the risky asset, the value of the risky asset, the amount contributed by the plan member and the value of the wealth process. Then, it is clear that we should investigate the dynamics

$$\begin{aligned} dX^{undisc, \pi^{undisc}}(t) &= \pi^{undisc}(t) \frac{dS^{undisc}(t)}{S^{undisc}(t)} + (X^{undisc, \pi^{undisc}}(t) - \pi^{undisc}(t))r(t)dt \\ &+ c^{undisc}(t)dt, \quad X^{undisc, \pi^{undisc}}(0) = x. \end{aligned}$$

By standard calculus we can now derive the dynamics (2.4) for the discounted value of the wealth process $X^\pi(t) = e^{-\int_0^t r(s)ds} X^{undisc, \pi^{undisc}}(t)$, which is controlled with a process π describing the discounted amount of money which should be invested in the risky asset.

We assume there is no agency problem, i.e. the pension plan manager and the pension beneficiary have the same optimization objective. The pension beneficiary is interested in maximizing the expected utility of the excess wealth over a pre-specified level at the retirement age. We neglect mortality risk, i.e. our problem is conditional on survival of the pension beneficiary throughout the contribution period.⁵ By considering the excess

⁵If we assume that the time of death of the beneficiary is independent of the financial market risk and the macroeconomic risk, then the mortality risk does not affect the optimal investment strategy. More specifically, let the time of death be modelled by a random variable τ which is independent of the financial market risk and the macroeconomic risk. The objective for a DC pension plan takes the form

$$\sup_{\pi \in \mathcal{A}} E[-e^{-\alpha(X^\pi(T)-F)} \mathbf{1}\{\tau \geq T\}],$$

wealth, we incorporate the fact that the pension beneficiary is interested in achieving a target payment which ensures his lifetime retirement. As the terminal pension fund might fall below the target level, the utility functions like log utility and power utility are inapplicable here since these functions are exclusively defined for the positive real line. In this paper we assume that the pension beneficiary is interested in maximizing the expected exponential utility of the discounted excess wealth:

$$\sup_{\pi \in \mathcal{A}} E[-e^{-\alpha(X^\pi(T)-F)}], \quad (2.5)$$

where $\alpha > 0$ is the risk aversion coefficient, and F denotes a target discounted payment for the pension beneficiary. Let us remark that the exponential utility is widely used and well motivated in economics, finance, insurance and risk management, see e.g. Carmona (2008). As far as the target is concerned we only assume that

(A4) F is \mathcal{F}_T -measurable, positive and bounded.

One possible example of the target payment F is

$$F = \kappa G(T)a(J(T)), \quad (2.6)$$

where κ is a fraction parameter, $G(T)$ denotes the discounted salary of the pension beneficiary at the time of retirement, a denotes an annuity factor for the lifetime annuity which depends on the future state of the economy $J(T)$, e.g. on the term structure of interest rates at the time of retirement. If the target (2.6) is chosen, then the pension beneficiary is interested in keeping his/her last salary income (or a fraction κ of the last salary income) as the lifetime annuity benefit. Note that maximizing the exponential utility of the excess wealth is in fact also related to minimizing the probability that the

which by the independence assumption is equal to

$$\sup_{\pi \in \mathcal{A}} E[-e^{-\alpha(X^\pi(T)-F)}]P(\tau \geq T).$$

terminal wealth $X(T)$ falls below the target level F .

Let us remark that in our model the family $\{e^{-\alpha X^\pi(\tau)}, \mathbb{F} - \text{stopping time } \tau \in [0, T]\}$ is uniformly integrable for any $\pi \in \mathcal{A}$. It is easy to notice that

$$\begin{aligned} e^{-\alpha X^\pi(\tau)} &= e^{-\alpha x - \alpha \int_0^\tau \pi(s)\mu(s)ds - \alpha \int_0^\tau c(s)ds - \alpha \int_0^\tau \pi(s)\sigma dW_1(s)} \\ &\leq K e^{-\int_0^\tau \alpha \pi(s)\sigma(s)dW_1(s) - \frac{1}{2} \int_0^\tau |\alpha \pi(s)\sigma(s)|^2 ds}, \quad 0 \leq \tau \leq T. \end{aligned} \quad (2.7)$$

The family $\{e^{-\int_0^\tau \alpha \pi(s)\sigma(s)dW_1(s) - \frac{1}{2} \int_0^\tau |\alpha \pi(s)\sigma(s)|^2 ds}, \mathbb{F} - \text{stopping time } \tau \in [0, T]\}$ is uniformly integrable by the Novikov criterion, see Theorem III.45 in Protter (2004). Hence, from (2.7) we conclude that $\{e^{-\alpha X^\pi(\tau)}, \mathbb{F} - \text{stopping time } \tau \in [0, T]\}$ is uniformly integrable for any $\pi \in \mathcal{A}$.

3 The solution to the investment problem

To solve our optimization problem (2.5), we rely on Backward Stochastic Differential Equations (BSDEs). We follow the approach from Hu et. al. (2005), Becherer (2006) and Chapter 11 in Delong (2013), to which the reader is referred to for further details.

In order to use the theory of BSDEs we assume that

(A5) every $(\mathbb{P}, \mathcal{F})$ local martingale M has the representation

$$\begin{aligned} M(t) &= M(0) \\ &+ \int_0^t \mathcal{Z}_1(s)dW_1(s) + \int_0^t \mathcal{Z}_2(s)dW_2(s) + \int_0^t \sum_{i=1}^I \mathcal{U}_i(s)d\tilde{N}_i(s), \quad 0 \leq t \leq T, \end{aligned}$$

with \mathbb{F} -predictable processes $(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{U}_1, \dots, \mathcal{U}_I)$ which are integrable in the Itô sense.

Let us remark that this assumption is satisfied if we define the probability space and the driving processes in an appropriate way, see Crépey (2011) for details.

We consider the BSDE

$$Y(t) = F + \int_t^T f(s)ds - \int_t^T Z_1(s)dW_1(s) - \int_t^T Z_2(s)dW_2(s) - \int_t^T \sum_{i=1}^I U_i(s)d\tilde{N}_i(s), \quad 0 \leq t \leq T, \quad (3.1)$$

where f is the generator of the equation which will be determined in the sequel. The solution to the BSDE (3.1) consists of square integrable processes $(Y, Z_1, Z_2, U_1, \dots, U_I)$ such that Y is \mathbb{F} -adapted and $(Z_1, Z_2, U_1, \dots, U_I)$ are \mathbb{F} -predictable. We introduce the process $A^\pi := (A^\pi(t), 0 \leq t \leq T)$ defined by

$$A^\pi(t) = -e^{-\alpha(X^\pi(t)-Y(t))}, \quad 0 \leq t \leq T, \pi \in \mathcal{A}.$$

The process A^π plays the key role in solving our optimization problem. It is straightforward to notice that

$$\mathbb{E}[-e^{-\alpha(X^\pi(T)-F)}] = \mathbb{E}[-e^{-\alpha(X^\pi(T)-Y(T))}] = \mathbb{E}[A^\pi(T)].$$

If A^π is a supermartingale for every $\pi \in \mathcal{A}$, then we obtain the inequality

$$\mathbb{E}[A^\pi(T)] = \mathbb{E}[-e^{-\alpha(X^\pi(T)-Y(T))}] \leq A^\pi(0), \quad \pi \in \mathcal{A}, \quad (3.2)$$

and if A^{π^*} is a martingale for some $\pi^* \in \mathcal{A}$, then we derive the equality

$$\mathbb{E}[A^{\pi^*}(T)] = \mathbb{E}[-e^{-\alpha(X^{\pi^*}(T)-Y(T))}] = A^{\pi^*}(0). \quad (3.3)$$

Combining (3.2) with (3.3), we get

$$\mathbb{E}[-e^{-\alpha(X^\pi(T)-Y(T))}] \leq \mathbb{E}[-e^{-\alpha(X^{\pi^*}(T)-Y(T))}], \quad \pi \in \mathcal{A},$$

and we conclude that the strategy π^* is optimal and $A^{\pi^*}(0)$ is the optimal value function of the optimization problem (2.5). Therefore, we aim to find the generator f^* of the BSDE (3.1), independent of π , such that the process A^π is a super-martingale for any $\pi \in \mathcal{A}$ and A^{π^*} is a martingale for some $\pi^* \in \mathcal{A}$.

We show how to find (f^*, π^*) . From (2.4) and (3.1) we get

$$\begin{aligned} -\alpha(X^\pi(t) - Y(t)) &= -\alpha(x - Y(0)) \\ &\quad -\alpha\left(\int_0^t (\pi(s)\mu(s) + c(s) + f(s))ds + \int_0^t (\pi(s)\sigma(s) - Z_1(s))dW_1(s) \right. \\ &\quad \left. - \int_0^t Z_2(s)dW_2(s) - \int_0^t \sum_{i=1}^I U_i(s)d\tilde{N}_i(s)\right), \quad 0 \leq t \leq T. \end{aligned}$$

We introduce two processes:

$$\begin{aligned} D^\pi(t) &= -\alpha\pi(t)\mu(t) - \alpha c(t) - \alpha f(t) + \frac{1}{2}\alpha^2(\pi(t)\sigma(t) - Z_1(t))^2 \\ &\quad + \frac{1}{2}\alpha^2(Z_2(t))^2 - \sum_{i=1}^I (\alpha U_i(t) - e^{\alpha U_i(t)} + 1)\lambda_i(t), \quad 0 \leq t \leq T, \\ M^\pi(t) &= \exp \left\{ -\int_0^t \alpha(\pi(s)\sigma(s) - Z_1(s))dW_1(s) - \int_0^t \frac{1}{2}\alpha^2(\pi(s)\sigma(s) - Z_1(s))^2 ds \right. \\ &\quad \left. + \int_0^t \alpha Z_2(s)dW_2(s) - \int_0^t \frac{1}{2}\alpha^2(Z_2(s))^2 ds \right. \\ &\quad \left. + \int_0^t \sum_{i=1}^I \alpha U_i(s)d\tilde{N}_i(s) + \int_0^t \sum_{i=1}^I (\alpha U_i(s) - e^{\alpha U_i(s)} + 1)\lambda_i(s) ds \right\}, \quad 0 \leq t \leq T, \end{aligned}$$

and we can write the candidate value function in the form:

$$A^\pi(t) = -e^{-\alpha(x-Y(0))} e^{\int_0^t D^\pi(s)ds} M^\pi(t), \quad 0 \leq t \leq T. \quad (3.4)$$

Since M^π is the stochastic exponential of a local martingale, the process M^π is a local martingale for any $\pi \in \mathcal{A}$. Let us recall that the value function derived from the optimal investment strategy should be a martingale, whereas with all the other strategies, the

value function should be a super-martingale. We choose

$$\begin{aligned}
\pi^*(t) &= \arg \min_{\pi \in \mathcal{A}} \left\{ -\alpha\pi\mu(t) + \frac{1}{2}\alpha^2(\pi\sigma(t) - Z_1(t))^2 \right\}, \quad 0 \leq t \leq T, \\
\alpha f^*(t) &= \min_{\pi \in \mathcal{A}} \left\{ -\alpha\pi\mu(t) + \frac{1}{2}\alpha^2(\pi\sigma(t) - Z_1(t))^2 \right\} - \alpha c(t) + \frac{1}{2}\alpha^2(Z_2(t))^2 \\
&\quad - \sum_{i=1}^I (\alpha U_i(t) - e^{\alpha U_i(t)} + 1)\lambda_i(t), \quad 0 \leq t \leq T.
\end{aligned} \tag{3.5}$$

With this choice we have $D^{\pi^*}(t) = 0$, $0 \leq t \leq T$, and $D^\pi(t) \geq 0$, $\pi \in \mathcal{A}$, $0 \leq t \leq T$. Moreover, f^* is independent of π .

We end up with the BSDE

$$\begin{aligned}
Y(t) &= F \\
&\quad + \int_t^T \left(\min_{\pi \in \mathcal{A}} \left\{ -\pi\mu(s) + \frac{1}{2}\alpha(\pi\sigma(s) - Z_1(s))^2 \right\} - c(s) + \frac{1}{2}\alpha(Z_2(s))^2 \right. \\
&\quad \left. - \frac{1}{\alpha} \sum_{i=1}^I (\alpha U_i(s) - e^{\alpha U_i(s)} + 1)\lambda_i(s) \right) ds \\
&\quad - \int_t^T Z_1(s) dW_1(s) - \int_t^T Z_2(s) dW_2(s) - \int_t^T \sum_{i=1}^I U_i(s) d\tilde{N}_i(s), \quad 0 \leq t \leq T.
\end{aligned} \tag{3.6}$$

In the last step, we transform the BSDE (3.6) with a non-Lipschitz generator to a BSDE with a Lipschitz generator. We introduce new variables:

$$\begin{aligned}
V(t) &= e^{\alpha Y(t)}, \quad P_1(t) = \alpha V(t-)Z_1(t), \quad P_2(t) = \alpha V(t-)Z_2(t), \\
Q_i(t) &= V(t-)e^{\alpha U_i(t)} - V(t-), \quad i = 1, \dots, I, \quad 0 \leq t \leq T.
\end{aligned} \tag{3.7}$$

Applying Itô's formula, we obtain the BSDE

$$\begin{aligned}
V(t) &= e^{\alpha F} \\
&+ \int_t^T \left(\min_{\pi \in \mathcal{A}} \left\{ -\alpha \pi \mu(s) V(s-) + \frac{1}{2} \alpha^2 \pi^2 \sigma^2(s) V(s-) - \alpha \pi \sigma(s) P_1(s) \right\} - \alpha c(s) V(s-) \right) ds \\
&- \int_t^T P_1(s) dW_1(s) - \int_t^T P_2(s) dW_2(s) - \int_t^T \sum_{i=1}^I Q_i(s) d\tilde{N}_i(s), \quad 0 \leq t \leq T,
\end{aligned}$$

which we use to characterize the optimal value function and the optimal investment strategy.

We present the main result of this paper.

Theorem 3.1. *Assume that (A1)-(A5) hold. Consider the BSDE*

$$\begin{aligned}
V(t) &= e^{\alpha F} \\
&+ \int_t^T \left(\min_{\pi \in \mathcal{A}} \left\{ -\alpha \pi \mu(s) V(s-) + \frac{1}{2} \alpha^2 \pi^2 \sigma^2(s) V(s-) - \alpha \pi \sigma(s) P_1(s) \right\} - \alpha c(s) V(s-) \right) ds \\
&- \int_t^T P_1(s) dW_1(s) - \int_t^T P_2(s) dW_2(s) - \int_t^T \sum_{i=1}^I Q_i(s) d\tilde{N}_i(s), \quad 0 \leq t \leq T. \tag{3.8}
\end{aligned}$$

a) *There exists a unique solution $(V, P_1, P_2, Q_1, \dots, Q_I)$ to the BSDE (3.8) such that V is \mathbb{F} -adapted and $(P_1, P_2, Q_1, \dots, Q_I)$ are \mathbb{F} -predictable. Moreover, V is strictly positive, bounded away from zero and from above, (Q_1, \dots, Q_I) are bounded $\mathbb{P} \otimes dt$ -a.e and (P_1, P_2) are square integrable.*

b) *The optimal value function of the utility maximization problem (2.5) is equal to $-e^{-\alpha x} V(0)$ and the optimal admissible investment strategy is given by*

$$\pi^*(t) = \max \left\{ K_1(t), \min \left\{ K_2(t), \frac{\mu(t)}{\alpha \sigma^2(t)} + \frac{P_1(t)}{\alpha \sigma(t) V(t-)} \right\} \right\}, \quad 0 \leq t \leq T.$$

Proof. a) Consider the function

$$f^\pi(s, v, p) = -\alpha\pi\mu(s)v + \frac{1}{2}\alpha^2\pi^2\sigma^2(s)v - \alpha\pi\sigma(s)p - \alpha c(s)v, \quad (s, v, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \pi \in \mathcal{A}.$$

Under our assumptions it is straightforward to conclude that $f^\pi(s, v, p)$ is Lipschitz continuous in (v, p) uniformly in (s, π) . Hence, $f^*(s, v, p) = \min_{\pi \in \mathcal{A}} f^\pi(s, v, p)$ is also Lipschitz continuous in (v, p) uniformly in s . Consequently, there exists a unique solution to the BSDE (3.8), see Proposition 3.2 in Becherer (2006) or Theorem 3.1.1 in Delong (2013). Since the generator f^* of the BSDE (3.8) is Lipschitz continuous, we can write

$$\begin{aligned} dV(t) = & -(L(t)V(t-) + H(t)P_1(t))dt \\ & + P_1(t)dW_1(t) + P_2(t)dW_2(t) + \sum_{i=1}^I Q_i(t)d\tilde{N}_i(t), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} L(t) &= \frac{f^*(t, V(t-), P_1(t)) - f^*(t, 0, P_1(t))}{V(t-)} \mathbf{1}\{V(t-) \neq 0\}, \quad 0 \leq t \leq T, \\ H(t) &= \frac{f^*(t, 0, P_1(t))}{P_1(t)} \mathbf{1}\{P_1(t) \neq 0\}, \quad 0 \leq t \leq T, \end{aligned}$$

and L and H are bounded. Let us define an equivalent probability measure with the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = e^{\int_0^T H(s)dW_1(s) - \frac{1}{2} \int_0^T (H(s))^2 ds}.$$

Changing the measure and taking the expectation, see Proposition 2.2 in El Karoui et al. (1997) or Propositions 3.3.1, 3.4.1 in Delong (2013), we can deduce from (3.9) the

representation

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{\alpha F} e^{\int_t^T L(s) ds} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (3.10)$$

The assertion concerning the solution V follows from boundedness of F and L . From the dynamics (3.8) we can also deduce that

$$\sum_{i=1}^I Q_i(t)(N_i(t) - N_i(t-)) = V(t) - V(t-), \quad 0 \leq t \leq T,$$

and we can conclude that each Q_i is bounded $\mathbb{P} \otimes dt$ -a.e. since only one counting process can jump at a time and V is bounded.

b) Since the BSDE (3.8) has a unique solution $(Y, P_1, P_2, Q_1, \dots, Q_I)$, the BSDE (3.6) has also a unique solution defined by

$$\begin{aligned} Y(t) &= \frac{1}{\alpha} \ln V(t), \quad Z_1(t) = \frac{P_1(t)}{\alpha V(t-)}, \quad Z_2(t) = \frac{P_2(t)}{\alpha V(t-)}, \\ U_i(t) &= \frac{1}{\alpha} \ln \left(\frac{Q_i(t) + V(t-)}{V(t-)} \right), \quad i = 1, \dots, I, \quad 0 \leq t \leq T, \end{aligned}$$

and Y is \mathbb{F} -adapted, bounded, (U_1, \dots, U_I) are \mathbb{F} -predictable, bounded $\mathbb{P} \otimes dt$ -a.e and (Z_1, Z_2) are \mathbb{F} -predictable, square integrable. It is obvious that $\pi^* \in \mathcal{A}$. We are left with proving the optimality principle for the process $A^\pi(t) = -e^{-\alpha(X^\pi(t) - Y(t))} = -e^{-\alpha(x - Y(0))} e^{\int_0^t D^\pi(s) ds} M^\pi(t)$ defined in (3.4). Since for any $\pi \in \mathcal{A}$ the process M^π is a positive local martingale (it is the stochastic exponential of a local martingale) and $D^\pi(t) \geq 0$, $0 \leq t \leq T$, we can derive

$$\begin{aligned} \mathbb{E}[A^\pi(t \wedge \tau_n) | \mathcal{F}_s] &= \mathbb{E} \left[-e^{-\alpha(x - Y(0)) + \int_0^{t \wedge \tau_n} D^\pi(u) du} M^\pi(t \wedge \tau_n) \middle| \mathcal{F}_s \right] \\ &\leq -e^{-\alpha(x - Y(0)) + \int_0^{s \wedge \tau_n} D^\pi(u) du} \mathbb{E} \left[M^\pi(t \wedge \tau_n) \middle| \mathcal{F}_s \right] \\ &= -e^{-\alpha(x - Y(0)) + \int_0^{s \wedge \tau_n} D^\pi(u) du} M^\pi(s \wedge \tau_n) = A^\pi(s \wedge \tau_n), \quad 0 \leq s \leq t \leq T, \end{aligned} \quad (3.11)$$

where $(\tau_n)_{n \geq 1}$ denotes a localizing sequence for the local martingale M^π . From the uniform

integrability of the family $\{e^{-\alpha X^\pi(\tau)}, \mathbb{F} - \text{stopping times } \tau \in [0, T]\}$ for $\pi \in \mathcal{A}$, see (2.7), and boundedness of Y we conclude that the family $\{A^\pi(\tau), \mathbb{F} - \text{stopping times } \tau \in [0, T]\}$ is uniformly integrable for $\pi \in \mathcal{A}$. Taking the limit $n \rightarrow \infty$ in (3.11), we obtain the supermartingale property of A^π for any $\pi \in \mathcal{A}$. For π^* we have $D^{\pi^*}(t) = 0$, $0 \leq t \leq T$, and we obtain the martingale property for A^{π^*} . For details we refer to Hu et. al. (2005), Becherer (2006) and Chapter 11.1 in Delong (2013). \square

We have characterized the optimal value function of our utility maximization problem (2.5) and the optimal investment strategy with the solution to the backward stochastic differential equation (3.8). The BSDE (3.8) is a non-linear BSDE with a Lipschitz generator. In our general model the solution to the BSDE cannot be found in a closed form and we have to derive the solution numerically. An efficient method to derive a solution to a BSDE is to apply Least Squares Monte Carlo which we discuss in Section 4. In one case the solution to the BSDE (3.8) has a nice closed-form representation.

Proposition 3.1. *Let the assumptions of Theorem 3.1 hold. If c, F, K_1, K_2 and λ are independent of W_1 , then $P_1(t) = 0$, $0 \leq t \leq T$, and*

$$\begin{aligned} \pi^*(t) &= \max \left\{ K_1(t), \min \left\{ K_2(t), \frac{\mu(t)}{\alpha \sigma^2(t)} \right\} \right\}, \quad 0 \leq t \leq T, \\ V(t) &= \mathbb{E} \left[e^{\alpha F} e^{\int_t^T \left(-\alpha \pi^*(s) \mu(s) + \frac{1}{2} \alpha^2 (\pi^*(s))^2 \sigma^2(s) - \alpha c(s) \right) ds} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \end{aligned}$$

Proof. Since the coefficients in front of V and P_1 in the generator of the BSDE (3.8) and the terminal condition for the BSDE (3.8) do not depend on the Brownian motion W_1 , we can choose $P_1(t) = 0$. The form of π^* is obvious. The representation of V now follows from (3.10). In this case the BSDE (3.8) is a linear BSDE. \square

Let us remark that the lack of dependence of c, F, K_1, K_2, λ on W_1 means that the contribution process, the target payment, the investment limits and the transition intensities are not related to the development of the risky asset S , i.e. they do not have a tradeable risk component.

Let us now comment on the optimal value function and the optimal investment strategy. We shall comment on the certainty equivalent (CE) instead of the optimal value function. The use of the CE makes the quantities easier to interpret, because the CE expresses the expected utility in monetary units instead of utility units. In our case we can define the certainty equivalent as an amount of a certain capital which a pension plan member should receive at time $t = 0$ as an equivalent for an uncertain terminal wealth which arises from the optimally invested contributions in the financial market. The equivalence of wealth is measured with the exponential utility of the discounted excess wealth over the target payment. Hence, the certainty equivalent CE solves the equation

$$\sup_{\pi \in \mathcal{A}} \mathbb{E} \left[-e^{-\alpha(X^\pi(T)-F)} \right] = \mathbb{E} \left[-e^{-\alpha(x+CE-F)} \right],$$

and by Theorem 3.1 we get

$$CE = -\frac{1}{\alpha} \ln(V(0)) + \frac{1}{\alpha} \ln \mathbb{E}[e^{\alpha F}]. \quad (3.12)$$

Notice that $x+CE$ can be interpreted as the certainty equivalent of the discounted amount which is annuitized by the pension plan member at the time of retirement T . We can also define the certainty equivalent for the excess wealth CE^{excess} as a solution to the equation

$$\sup_{\pi \in \mathcal{A}} \mathbb{E} \left[-e^{-\alpha(X^\pi(T)-F)} \right] = -e^{-\alpha(x+CE^{excess})},$$

and we get

$$CE^{excess} = -\frac{1}{\alpha} \ln(V(0)). \quad (3.13)$$

The optimal investment strategy π^* consists of two parts. The first part $\mu(t)/(\alpha\sigma^2(t))$ is the Merton investment strategy, which is the optimal investment strategy for an investor who aims to maximize the expected exponential utility of the terminal wealth in a one-

state economy without a contribution process, a target payment and investment limits. The second part of the optimal investment strategy $P_1(t)/(\alpha\sigma(t)V(t-))$ is used by the pension fund manager, who trades the risky asset S , to hedge the tradeable component in the contribution process, the target payment, the transition intensities and the investment limits. Since (3.7) holds, we have $P_1(t)/(\alpha\sigma(t)V(t-)) = Z_1(t)/\sigma(t)$ where Z_1 is the control process of the BSDE (3.6). From the theory of BSDEs, see Corollary 4.1 in El Karoui et al. (1997) or Theorem 4.1.4 in Delong (2013), we can deduce that the process Z_1 defines the change in the value of the process Y resulting from changes in the risky asset S due to the movement of the Brownian motion W_1 . Since (3.7) and (3.13) hold, we can conclude that the second part of the optimal investment strategy is used by the pension fund manager to follow the opposite changes in the certainty equivalent for the excess wealth resulting from changes in the contribution process, the target payment, the transition intensities and the investment limits due to the tradeable risk component W_1 .

4 Numerical example

This section reports some numerical results based on the model developed in previous sections. We consider a two-state Markov-regime-switching model. State 1 denotes economic boom and state 2 denotes economic recession. The economy is in state 1 at time $t = 0$. The salary income of a pension plan member is modelled by the stochastic differential equation

$$dG(t) = \mu_G(J(t-))G(t)dt + \sigma_G(J(t-))G(t)(\rho dW_1(t) + \sqrt{1 - \rho^2}dW_2(t)), \quad G(0) = g,$$

and we choose a proportional contribution rate, i.e.

$$c(t) = \gamma G(t), \quad 0 \leq t \leq T.$$

T	$S(0)$	$G(0)$	γ	K_1	K_2	ρ	x	α
1	100	10	0.1	0	60	0.5	200	0.1

Table 1: The values of the state-independent parameters.

The initial capital in the pension fund is

$$x = G(0)a(1),$$

where $a(1)$ denotes a lifetime annuity factor calculated based on macroeconomic assumptions for state 1 (e.g. based on the term structure of interest rates in economic boom). The choice of the initial capital implies that if the pension fund is annuitized then it can be converted to a lifetime annuity with the benefit equal to the current salary of the pension beneficiary. The target payment for the pension plan is of the form

$$F = G(T)a(J(T)),$$

hence the pension beneficiary is interested in a lifetime annuity with the benefit equal to his/her last salary.

We fix the parameters for the numerical example as in Tables 1-2. We assume that short-selling of the risky asset is prohibited for the pension fund and the upper bound for the investment strategy is 150% of the Merton optimal investment strategy in state 1 for the exponential utility ($K_2 = 1.5 * \mu(1)/(\sigma^2(1)\alpha) = 60$). We choose a moderate correlation ρ between the risky asset and the salary process. It is reasonable to assume that the drift is higher in state 1 than in state 2, and the volatility is higher in state 2 than in state 1. The lifetime annuity factor is higher in state 2 than in state 1 (e.g. due to the fall of interest rates in recession). In order to guarantee that the contribution process is bounded we introduce an upper bound on the contribution and we redefine $c(t) = \min\{c(t), 20\}$.

state i	$\mu(i)$	$\sigma(i)$	$\mu_G(i)$	$\sigma_G(i)$	$\lambda(i)$	$a(i)$
state 1	0.04	0.10	0.03	0.02	1	20
state 2	0.01	0.20	0	0.06	2	22

Table 2: The values of the state-dependent parameters.

We would like to compare the certainty equivalents (3.12), the excess wealth $X^{\pi^*}(T) - F$ and the replacement ratios $X^{\pi^*}(T)/F$ under the optimal investment strategy π^* in different scenarios. We have to solve the BSDE (3.8). Let us comment how the solution (V, P_1) , which we need to define the optimal investment strategy and the certainty equivalent, can be derived numerically, see Chapter 5.1 in Delong (2013) for details. First, we introduce a partition $0 = t_0 < t_1 < \dots < t_i < \dots < t_n = T$ of the time interval $[0, T]$ with a time step h . Next, the solution can be defined by the recursive relation:

$$\begin{aligned}
V_j(T) &= e^{\alpha G(T)a(j)}, \quad j = 1, 2, \\
P_{1,j}(t_i) &= \frac{1}{h} \mathbb{E} \left[V_{J(t_{i+1})}(t_{i+1}) (W_1(t_{i+1}) - W_1(t_i)) | G(t_i) = g, J(t_i) = j \right], \quad j = 1, 2, \quad i = 0, \dots, n-1, \\
V_j(t_i) &= \mathbb{E} \left[V_{J(t_{i+1})}(t_{i+1}) \right. \\
&\quad \left. + \left(\min_{0 \leq \pi \leq K_2} \left\{ -\alpha \pi \mu(j) V_{J(t_{i+1})}(t_{i+1}) + \frac{1}{2} \alpha^2 \pi^2 \sigma^2(j) V_{J(t_{i+1})}(t_{i+1}) - \alpha \pi \sigma(j) P_{1,j}(t_i) \right\} \right. \right. \\
&\quad \left. \left. - \alpha \gamma g V_{J(t_{i+1})}(t_{i+1}) \right) h | G(t_i) = g, J(t_i) = j \right], \quad j = 1, 2, \quad i = 0, \dots, n-1, \quad (4.1)
\end{aligned}$$

see Bouchard and Elie (2008). Finally, the expectations in (4.1) are estimated by the Least Squares Monte Carlo method, i.e. are estimated by fitting regression polynomials at each point $(t_i)_{i=0, \dots, n-1}$ with a dependent variable $G(t_i)$ based on a generated sample of $(G(t_i), J(t_i))_{i=1, \dots, n}$, see Longstaff and Schwartz (2001).

In Tables 3-4 we find the certainty equivalents $CE = -\frac{1}{\alpha} \ln(V(0)) + \frac{1}{\alpha} \ln \mathbb{E}[e^{\alpha F}]$, the expected excess wealth $\mathbb{E}[X^{\pi^*}(T) - F]$ and the expected replacement ratios $\mathbb{E}[X^{\pi^*}(T)/F]$ computed for diverse parameter combinations. We have the following observations:

- The higher the investment limit K_2 or the higher the fraction γ of the salary contributed into the pension fund, the higher the certainty equivalent, the expected

	$\mu_G(1) = 0.03$	$\mu_G(1) = 0.04$	$\mu_G(1) = 0.06$
CE	3.722	3.706	3.688
	$\sigma_G(2) = 0.04$	$\sigma_G(2) = 0.06$	$\sigma_G(2) = 0.1$
CE	3.043	3.722	7.462
	$\lambda(2) = 1$	$\lambda(2) = 2$	$\lambda(2) = 3$
CE	3.675	3.722	3.788
	$a(2) = 20$	$a(2) = 22$	$a(2) = 25$
CE	3.312	3.722	4.386
	$K_2 = 30$	$K_2 = 60$	$K_2 = 100$
CE	3.060	3.722	3.851
	$\gamma = 0.1$	$\gamma = 0.2$	$\gamma = 0.3$
CE	3.722	4.849	5.875
	$\rho = 0.1$	$\rho = 0.5$	$\rho = 0.9$
CE	2.149	3.722	6.081
	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.15$
CE	3.257	3.722	7.396

Table 3: The certainty equivalents under different parameter combinations. Other parameters are specified in Tables 1-2.

excess wealth and the expected replacement ratio. This conclusion is obvious. If the pension plan manager has less constraints on the investment policy and more contributions to invest, then he can apply the strategy which is closer to the optimal unconstrained strategy, he can optimally invest more funds in the market and he can achieve a higher terminal wealth (the target payment is not affected in this scenario).

- The higher the transition intensity $\lambda(2)$ from economic recession to economic boom, the higher the certainty equivalent, the expected excess wealth and the expected replacement ratio. We can notice that a higher transition intensity $\lambda(2)$ implies that the economy recovers faster from the recession and, consequently, the pension plan manager receives higher contributions and earns higher returns on the pension fund over longer time periods. Moreover, under a higher transition intensity $\lambda(2)$ there is a lower probability that the economy will be in recession at the time of retirement

	$\mu_G(1) = 0.03$	$\mu_G(1) = 0.04$	$\mu_G(1) = 0.06$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-8.140	-9.766	-13.043
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.964	0.956	0.942
	$\sigma_G(2) = 0.04$	$\sigma_G(2) = 0.06$	$\sigma_G(2) = 0.1$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-8.145	-8.140	-8.404
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.963	0.964	0.964
	$\lambda(2) = 1$	$\lambda(2) = 2$	$\lambda(2) = 3$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-10.127	-8.140	-6.714
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.955	0.964	0.970
	$a(2) = 20$	$a(2) = 22$	$a(2) = 25$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-1.635	-8.140	-17.919
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.993	0.964	0.928
	$K_2 = 30$	$K_2 = 60$	$K_2 = 100$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-9.151	-8.140	-7.706
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.959	0.964	0.966
	$\gamma = 0.1$	$\gamma = 0.2$	$\gamma = 0.3$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-8.140	-7.079	-6.067
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.964	0.969	0.974
	$\rho = 0.1$	$\rho = 0.5$	$\rho = 0.9$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-8.392	-8.140	-8.091
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.961	0.964	0.964
	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.15$
$\mathbb{E}[X^{\pi^*}(T) - F]$	-7.874	-8.140	-8.576
$\mathbb{E}[X^{\pi^*}(T)/F]$	0.964	0.964	0.961

Table 4: The expected excess wealth and the expected replacement ratios under different parameter combinations. Other parameters are specified in Tables 1-2.

and the terminal wealth is more likely to be compared with the target payment contingent on the lower annuity $a(1)$. However, a higher transition intensity $\lambda(2)$ also implies that the target payment which is contingent on the salary process is likely to be higher since the salary processes rises in the economic boom. We observe that the increase in the contributions and the returns of the pension fund in the economic boom compensate the increase in the target payment.

- The higher the annuity factor $a(2)$, the lower the expected excess wealth and the expected replacement ratio. This agrees with intuition since the terminal wealth is compared with a higher target payment. However, it is important to realize that in our model the target payment affects the optimal investment strategy since π^* depends on the solution (V, P_1) to the BSDE (3.8) with the terminal condition F . We can conclude that a higher target payment F , which can be partially hedged with the risky asset, forces the pension plan manager to invest more in the risky asset in order to achieve a higher terminal wealth and compensate the pension beneficiary for the decrease in the utility resulting in a higher target payment. With such an interpretation we can now justify the observation that the higher the annuity factor $a(2)$ is, the higher the certainty equivalent is. In spite of a higher wealth achieved by the pension plan manager, the higher target payment decreases the expected excess wealth and the expected replacement ratio as already noticed.
- The higher the volatility σ_G of the salary process, the higher the certainty equivalent. A higher volatility σ_G increases the chance that more contributions will be invested in the pension fund, the terminal wealth will be higher and the target payment will be higher. Using the interpretations we previously deduced, we can conclude that the certainty equivalent should indeed increase in σ_G . On the other side, notice that a higher volatility σ_G also increases the chance that less contributions will be invested in the pension fund, the terminal wealth will be lower and the target payment will be lower. The entire effect of the volatility σ_G of the salary process on

the expected excess wealth and the expected replacement ratio is negligible.

- The higher the correlation coefficient ρ between the tradeable risky asset and non-tradeable salary process, the higher the certainty equivalent, the expected excess wealth and the expected replacement ratio. The stronger the risky asset is correlated with the salary process, the easier it is for the pension plan manager to replicate the target payment contingent on the salary process and to achieve a higher excess wealth, and most importantly, a higher terminal wealth. The impact of the correlation coefficient on the expected excess wealth and the expected replacement ratio is small. However, it is also worth pointing out that the higher the correlation coefficient ρ is, the lower the variance of the excess wealth is. The standard deviations for the excess wealth are 12.049, 11.423, 9.740 for $\rho = 0.1, 0.5, 0.9$. This observation indicates an obvious conclusion that the investment portfolio hedges the target payment more effectively if ρ is higher.
- The higher the risk aversion coefficient α , the lower the expected excess wealth and the expected replacement ratio. This pattern agrees with intuition. The more risk averse the pension plan manager, the smaller amount he invests in the risky asset, and consequently, he is likely to achieve a lower terminal wealth (the target payment is not affected in this scenario). The higher the risk aversion coefficient α , the higher the certainty equivalent. This is surprising at the first sight in the view of the fact that the expected terminal wealth under the optimal investment strategy decreases in α (since the amount invested in the risky asset is lower for a more risk averse manager). This is true for a given target payment F . However, in order to achieve the optimal value function $V(0)$, which is used to define the certainty equivalent, the pension plan manager optimally adjusts the investment strategy to follow the target payment F , whereas the certainty equivalent is not invested in the market. Hence, it is more likely that a shortfall in the excess wealth $CE - F$ arises, which is more severe in terms of the utility for higher risk aversion coefficients. In order

to compensate a more severe shortfall in the excess wealth $CE - F$ for higher risk aversion coefficients, a higher certainty equivalent is required for higher α .

- The higher the drift μ_G of the salary process, the lower the expected excess wealth and the expected replacement ratio. The higher drift μ_G implies that the amounts contributed to the pension fund are likely to be higher and the target payment, which is contingent on the salary process, is likely to be higher. It seems that the increase in the contributions does not compensate the increase in the target payment. Hence, the expected excess wealth and the expected replacement ratio decreases in μ_G . We also observe that the higher the drift μ_G of the salary process, the lower the certainty equivalent. This is not clear in the view of the fact that the expected terminal wealth under the optimal investment strategy increases in μ_G (since more contributions flow into the pension fund). Yet, the certainty equivalent does not solely depend on the expected terminal wealth under the optimal investment strategy, it also depends on the target payment. As previously noticed, the excess wealth decreases in μ_G and, consequently, the optimal value $V(0)$ increases in μ_G . The pay-off $\mathbb{E}[e^{\alpha F}]$ increases in μ_G and the total outcome is that the certainty equivalent $-\frac{1}{\alpha} \ln(V(0)) + \frac{1}{\alpha} \ln \mathbb{E}[e^{\alpha F}]$ decreases in μ_G . We remark that the impact of the drift of the salary process on the certainty equivalent is small.

5 Conclusion

This paper looks into the investment behavior of a defined contribution pension plan in an economy with macroeconomic risks. We have considered an economy which can be in one of I states and switches randomly between those states. We have found the optimal investment strategy which maximizes the expected exponential utility of the discounted excess wealth over a target payment, e.g. a target lifetime annuity. The optimal value function of our optimization problem (the certainty equivalent) and the optimal invest-

ment strategy are the solution to a backward stochastic differential equation.

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References

- Asprem, M. (1989) Stock prices, asset portfolios and macroeconomic variables in ten European countries. *Journal of Banking and Finance* **13**, 589-612.
- Becherer, D. (2006) Bounded solutions to BSDE's with jumps for utility optimization and indifference hedging. *The Annals of Applied Probability* **16**, 2027-2054.
- Blake, D., Wright, D., and Zhang, Y.M. (2012) Target-driven investing: Optimal investment strategies in defined contribution pension plans under loss aversion. *Journal of Economic Dynamics and Control* **37**, 195-209.
- Bouchard, B., and Elie, R. (2008) Discrete time approximation of decoupled forward backward SDE with jumps. *Stochastic Processes and their Applications* **118**, 53-75.
- Boulier, J.F., Huang, S.J., and Taillard, G. (2001) Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics* **28**, 173-189.
- Browne, S. (1995) Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research*, 20(4), 937-958.
- Cairns, A.J.G., Blake, D., and Dowd, K. (2006) Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control* **30**, 843-877.

- Campbell, J.Y. (1987) Stock Returns and the Term Structure. *Journal of Financial Economics* **18**, 373-399.
- Carmona, R. (2008) *Indifference Pricing: Theory and Applications*. Princeton University Press.
- Crépey, S. (2011) About the pricing equation in finance. In *Paris-Princeton Lectures in Mathematical Finance 2010* (ed. Carmona, R.A., Çinlar, E., Ekeland, I., Jouini, E., Scheinkman, J.A., and Touzi, N.), pp. 63-203. Springer. Berlin.
- Delong, L. (2013) *Backward Stochastic Differential Equations with Jumps and their Actuarial and Financial Applications*. Springer. London.
- El Karoui, N., Peng, S., and Quenez, M.C. (1997) Backward stochastic differential equations in finance. *Mathematical Finance* **7**, 1-71.
- Elliott, R.J., and Siu, T.K., Badescu, A. (2011) On pricing and hedging options in regime-switching models with feedback effect. *Journal of Economic Dynamics and Control* **35**, 694-713.
- Engle, R., Ghysels, E., and Sohn, B. (2008) On the economic sources of stock market volatility. *Working paper*.
- Gao, J. (2008) Stochastic optimal control of dc pension funds. *Insurance: Mathematics and Economics* **42**, 1159-1164.
- Hardy, M.R. (2001) A regime-switching model of long-term stock returns. *North American Actuarial Journal* **5**, 41-53.
- Hu, Y., Imkeller, P., and Müller, M. (2005) Utility maximization in incomplete markets. *The Annals of Applied Probability* **15**, 1691-1712.
- Korn, R., Siu, T.K., and Zhang, A. (2011) Asset allocation for a DC pension fund under regime switching environment. *European Actuarial Journal* **1**, 361-377.

- Longstaff, F., and Schwartz, E. (2001) Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies* **14**, 113-147.
- Merton, R. (1969) Lifetime portfolio selection under uncertainty: the continuous-time case. *Review of Economic Statistics* **51**, 247-257.
- Merton, R. (1971) Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* **3**, 373-413.
- Norberg, R. (2001) On Bonus and Bonus Prognoses in Life Insurance. *Scandinavian Actuarial Journal* **2**, 126-147.
- Protter, P. (2004) *Stochastic Integration and Differential Equations*. Springer. Berlin.
- Schwert, G. (1989) Why does stock market volatility change over time? *Journal of Finance* **44**, 1115-1153.

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