

# Applications of Backward Stochastic Differential Equations to insurance and finance

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## Abstract

In this paper we deal with backward stochastic differential equations and give examples of their applications to insurance and finance. There are two major fields of applications. The first area concerns pricing and risk measures, the second deals with optimal control problems and optimization. Our aim is to show that backward stochastic differential equations, despite its mathematical complexity, are intuitive and can help in solving real life problems.

**Keywords:** backward stochastic differential equation, Choquet expectation,  $f$ -expectation, market consistent valuation, quadratic hedging, unit-linked products.

W pracy tej rozważamy wsteczne stochastyczne równania różniczkowe oraz ich zastosowania w ubezpieczeniach i finansach. Dwa główne obszary zastosowań obejmują: wycenę zobowiązań i miary ryzyka oraz problemy optymalizacyjne w teorii sterowania stochastycznego. Naszym celem jest pokazanie, że wsteczne stochastyczne równania różniczkowe, mimo iż wymagają znajomości zaawansowanego matematycznego aparatu, mogą być pomocne przy rozwiązaniu rzeczywistych problemów.

**Słowa kluczowe:** wsteczne stochastyczne równania różniczkowe, wartość oczekiwana Choquet'a,  $f$ -oczekiwana wartość, wycena rynkowa, kwadratowe zabezpieczenie, produkt na życie z funduszem inwestycyjnym.

# 1 Introduction

In this section we give a short introduction to backward stochastic differential equations and give a motivation for studying them. We would like to point out that the theory of backward stochastic differential equations, together with its applications, requires rigorous mathematical treatment which is omitted in this paper. Our aim is only to present a general idea and intuition without touching mathematical details.

The concept of ordinary differential equations (ODEs) is well-known. We encounter ordinary differential equations with a given initial condition  $\tilde{y}$

$$\frac{dy(t)}{dt} = f(t, y(t)), \quad y(0) = \tilde{y}, \quad (1.1)$$

or ordinary differential equations with a terminal condition  $\tilde{y}$

$$\frac{dy(t)}{dt} = -f(t, y(t)), \quad y(T) = \tilde{y}. \quad (1.2)$$

In general, both types of this equations can be solved by applying similar techniques and there is no difference, from the mathematical point, whether an initial condition or a terminal condition is provided. In the case when  $f$  is globally Lipschitz in  $y$  uniformly in  $t$ , Picard iterations can be used to show that the equation (1.1) or (1.2) has a unique solution  $y := (y(t), 0 \leq t \leq T)$ , which is a curve in  $\mathbb{R}$ , given by the integral representation

$$y(t) = \tilde{y} + \int_0^t f(s, y(s)) ds, \quad 0 \leq t \leq T,$$
$$y(t) = \tilde{y} + \int_t^T f(s, y(s)) ds, \quad 0 \leq t \leq T.$$

In applications ordinary differential equations usually describe a system and its evolution over time. The equation (1.1) or (1.2) gives the dynamics of a system under consideration. Ordinary differential equations have numerous applications in various fields, including insurance and finance. As an example of an ODE with an initial condition, let us mention Kolmogorov's equations which describe the evolution over time of transition probabilities in a multiple state insurance model. As far as an ODE with a terminal condition is concerned, recall Thiele's differential equation describing the evolution of a reserve of a life insurance product.

Before we move further, let us first introduce a probability space  $(\Omega, \mathbb{F}, \mathbb{P})$  and a filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  which captures all information available to an observer at time  $t$ . We assume that all processes are defined on  $(\Omega, \mathbb{F}, \mathbb{P})$  and are  $\mathbb{F}$ -adapted, in the sense that the value of a process at time  $t$  is known to an observer at time  $t$ .

In the theory of stochastic processes one investigates stochastic differential equations (SDEs) with an initial condition. Intuitively, SDEs extend ODEs (1.1) by

adding a random noise. The most common form of a stochastic differential equation is the following

$$dY(t) = f(t, Y(t))dt + \sigma(t, Y(t))dW(t), \quad Y(0) = \tilde{y}, \quad (1.3)$$

where a driving noise  $W := (W(t), 0 \leq t \leq T)$  denotes a Brownian motion. As before, such equations describe the dynamics of a system, but contrary to (1.1), the future evolution is unknown and depends on the randomness of the driving noise  $W$ . Given an initial condition, provided that we know from which state a system departs, we can solve a SDE. In the case when  $f, \sigma$  are globally Lipschitz continuous in  $y$  uniformly in  $(t, \omega)$ , the stochastic differential equation (1.3) has a unique, adapted solution  $(Y(t), 0 \leq t \leq T)$  which is a random process with the integral representation

$$Y(t) = \tilde{y} + \int_0^t f(s, Y(s))ds + \int_0^t \sigma(s, Y(s))dW(s), \quad 0 \leq t \leq T.$$

For details concerning SDEs we refer the reader to Chapter 5 in [9]. We remark that stochastic differential equations have found many applications in the modern insurance and financial mathematics. We just recall that the insurance surplus process in the insurance risk theory or the dynamics of a risky stock in Black-Scholes financial model are given by stochastic differential equations.

Finally, we can consider backward stochastic differential equations (BSDEs) driven by a Brownian motion with a given terminal condition  $\xi$

$$dY(t) = -f(t, Y(t), Z(t))dt + Z(t)dW(t), \quad Y(T) = \xi, \quad (1.4)$$

where  $\xi$  is a  $\mathcal{F}_T$ -measurable random variable, i.e. a variable whose value is known to an observer at the terminal time  $T$ . Notice that the extension of stochastic differential equations from the forward case (1.3) to the backward case (1.4) is much more difficult than for ordinary differential equations. When we solve an equation backwards, then intuitively, the values of a solution should depend on the value of a terminal condition. If a terminal condition is a constant, as it is the case for ODEs, then no problem arises, but when a terminal value is a  $\mathcal{F}_T$ -measurable random variable, as it is the case for BSDEs, then solving (1.4) in a naive way would give a solution whose values at intermediate times are known at the terminal  $T$ . We can see now that constructing an  $\mathbb{F}$ -adapted solution of (1.4) is not a trivial task. It turns out that a solution of the backward stochastic differential equation (1.4) is an  $\mathbb{F}$ -adapted pair  $(Y(t), Z(t), 0 \leq t \leq T)$  satisfying

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T.$$

Intuitively, the integral term  $\int ZdW$  has to be "subtracted" in order to get an adapted solution.

Backward stochastic differential equations have been introduced to the literature in 1990 by Pardoux and Peng, see [13]. Their key theorem is the following.

**Theorem 1.1.** *Under the following assumptions:*

1.  $\mathbb{E}[|\xi|^2] < \infty$ ,
2.  $\mathbb{E}[\int_0^T |f(s, 0, 0)|^2 ds] < \infty$ ,
3. *the function  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is product measurable,  $\mathbb{F}$ -adapted and Lipschitz continuous globally in  $(y, z)$  uniformly in  $(t, \omega)$ , i.e.*

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq K(|y - y'| + |z - z'|),$$

*holds for a.e.  $(\omega, t) \in \Omega \times [0, T]$  and any  $(y, z), (y', z')$  in  $\mathbb{R}^2$ ,*

*there exists a unique solution  $(Y, Z)$  of the backward stochastic differential equation (1.4), such that*

$$\mathbb{E}[\sup_{t \in [0, T]} |Y(t)|^2] < \infty, \quad \mathbb{E}[\int_0^T |Z(t)|^2 dt] < \infty.$$

*Moreover the solution satisfies the fixed point equation*

$$Y(t) = \mathbb{E}\left[\xi + \int_t^T f(s, Y(s), Z(s)) ds \middle| \mathcal{F}_t\right], \quad 0 \leq t \leq T.$$

The way to prove this theorem is to apply a martingale representation theorem and Banach fixed point theorem in appropriate spaces with adequate norms.

The second key theorem in the theory of BSDEs, from which many results follow, is a comparison principle. It has been proved in [11].

**Theorem 1.2.** *Consider the backward stochastic differential equations (1.4) with functions  $f, \tilde{f}$  and corresponding terminal values  $\xi, \tilde{\xi}$  satisfying the assumptions in Theorem 1.1. Let  $(Y, Z)$  and  $(\tilde{Y}, \tilde{Z})$  denote the associated unique solutions. Suppose that*

- $\xi \geq \tilde{\xi}$ , a.s.,
- $f(t, y, z) - \tilde{f}(t, y, z) \geq 0$ , a.e.  $(\omega, t) \in \Omega \times [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}$ .

*Then,  $Y(t) \geq \tilde{Y}(t)$  holds a.s. on  $[0, T]$ .*

This theorem is of great importance both for theoretical considerations and applications. The reader may recall that comparison theorems exist also in the case of ordinary differential equations and stochastic differential equations.

There are two major fields of applications of backward stochastic differential equations to insurance and finance. They can be applied to define prices of claims or to solve optimization problems. In Section 2 we consider pricing mechanisms generated by backward stochastic differential equations and replicating strategies derived from BSDEs. In Section 3 we comment on usefulness of backward stochastic differential equations in solving complex optimization problems and give an illustrative example of a quadratic hedging.

The reader interested in the theory of BSDEs is referred to [8] and [6] and references therein.

## 2 Pricing mechanisms, f-expectations and replicating strategies

The classical question in insurance and finance is how to value a risky position which provides a random endowment  $\xi$ . The Von Neumann and Morgenstein's axiomatic system of expected utilities suggests the following answer: every investor takes decisions based on his/her utility function and value a given position as if he/she is indifferent with respect to taking this position or not. In terms of mathematics, the premium  $P$  for the risk  $\xi$ , which an investor with a utility  $u$  should charge for insuring this risk, solves the equation

$$\mathbb{E}[u(X + P - \xi)] = \mathbb{E}[u(X)], \quad (2.1)$$

where  $X$  is an initial position. The most common utilities are linear and exponential, which give the following prices

$$\begin{aligned} u(x) = ax + b &\Rightarrow P = \mathbb{E}[\xi], \\ u(x) = -\frac{1}{\alpha}e^{-\frac{1}{\alpha}x} &\Rightarrow P = \alpha \log \mathbb{E}\left[e^{\frac{1}{\alpha}\xi}\right] \end{aligned}$$

However, Allais's paradox is well-known in the utility theory. This paradox shows that taking decisions based on (2.1) may lead to inconsistencies. One of the reasons why the utility theory fails is the additivity of the mathematical expected value or the additivity of probability:

$$\mathbb{E}[\alpha\xi + \beta\eta] = \alpha\mathbb{E}[\xi] + \beta\mathbb{E}[\eta].$$

The question arises is it possible to define a pricing operator which would give a price for a given risk and which would preserve much of the properties of the standard

expected value and conditional expected value except linearity. We remark that for applications one should require that such operator preserves not only properties of the unconditional expected values but also of the conditional expected values. The reason is that we not only want to find a price at a give initial time but we also want to observe how this price changes until a liability is covered.

One way to get rid of the additivity is to consider Choquet expectations. Recall that for a random variable  $\xi$  we define its expected value as

$$\mathbb{E}[\xi] = \int_{-\infty}^0 (Pr(\xi > t) - 1)dt + \int_0^{\infty} Pr(\xi > t)dt.$$

The idea of Choquet was to replace additive probability  $Pr$  by a non-additive function  $V$  which he called a capacity. We can define the Choquet expectation of  $\xi$  with respect to a capacity  $V$ , as

$$C[\xi] = \int_{-\infty}^0 (V(\xi > t) - 1)dt + \int_0^{\infty} V(\xi > t)dt. \quad (2.2)$$

Choquet expectations have found many applications to insurance and finance. In 1987 Yaari, see [15], proposed a theory of decision making under uncertainty which used Choquet expectations and the related notion of distorted probabilities. Similarly, in [14] distorted risk measures (later called Wang's premium principles) were introduced by Wang. However, Choquet expectations have a serious drawback as it is difficult to define conditional Choquet expectations in terms of Choquet expectations.

In 1997, Peng in [11] came out with the notion of f-expectations.

**Definition 2.1.** *Let  $\xi$  describes a risk we want to valuate. We define f-expectation and condition f-expectation of  $\xi$  as*

$$\mathcal{E}_f[\xi] = Y(0),$$

$$\mathcal{E}_f[\xi|\mathcal{F}_t] = Y(t), \quad 0 \leq t \leq T,$$

where  $Y$  is a solution of the backward stochastic differential equation with the generator  $f$  and the terminal condition  $\xi$

$$dY(t) = -f(t, Y(t), Z(t))dt + Z(t)dW(t), \quad Y(T) = \xi. \quad (2.3)$$

In order to have existence and uniqueness it is assumed that  $\xi$  and  $f$  satisfy the assumptions of Theorem 1.1, together with  $f(t, y, 0) = 0$ .

It turns out that the pricing operator, defined above, fulfills many desirable requirements, including properties satisfied by the classical mathematical (unconditional and conditional) expected value. In particular, we have:

1.  $\xi^1 \geq \xi^2 \Rightarrow \mathcal{E}_f(\xi^1|\mathcal{F}_t) \geq \mathcal{E}_f(\xi^2|\mathcal{F}_t), 0 \leq t \leq T,$
2.  $\mathcal{E}_f(\mathcal{E}_f(\xi|\mathcal{F}_s)|\mathcal{F}_t) = \mathcal{E}_f(\xi|\mathcal{F}_t), 0 \leq t \leq s \leq T,$
3.  $\mathcal{E}_f(c|\mathcal{F}_t) = c, 0 \leq t \leq T$
4.  $\mathcal{E}_f(\xi + c|\mathcal{F}_t) = \mathcal{E}_f(\xi|\mathcal{F}_t) + c, 0 \leq t \leq T \Leftrightarrow f$  is independent of  $y,$
5.  $\mathcal{E}_f(\lambda\xi|\mathcal{F}_t) = \lambda\mathcal{E}_f(\xi|\mathcal{F}_t), \lambda > 0, 0 \leq t \leq T$   
 $\Leftrightarrow f(t, \lambda y, \lambda y) = \lambda f(t, y, z), 0 \leq t \leq T,$
6.  $f(t, \lambda y + (1 - \lambda)y, \lambda z + (1 - \lambda)z) \leq \lambda f(t, y, z) + (1 - \lambda)f(t, y, z)$   
 $\Rightarrow \mathcal{E}_f(\lambda\xi^1 + (1 - \lambda)\xi^2|\mathcal{F}_t) \leq \lambda\mathcal{E}_f(\xi^1|\mathcal{F}_t) + (1 - \lambda)\mathcal{E}_f(\xi^2|\mathcal{F}_t), 0 \leq t \leq T.$

The proofs of this properties the reader can find in [1].

Above given relations have their financial interpretations. Point 1 means that the price for a greater risk is greater, point 3 states that the price for covering a non-risky constant equals that constant, from point 4 and 5 we conclude that the price for a risk increased in the additive or the multiplicative way by a non-risky constant equals the price for a risk plus that constant or multiplied by that constant, point 6 means that the price for diversified risk is smaller than for not diversified risk. Finally, point 2 may be interpreted in terms of dynamic programming principle which states than in order to value a risk at time 0 we can first value it at some intermediate date  $0 < t < T$  and then value it again from time  $t$  to time 0. Notice that points 5 and 6 hold provided that the function  $f$  satisfies some additional assumptions.

We can conclude that the form of the generator  $f$  defines the pricing mechanism in a market or risk aversion of an investor. As in the utility theory where in order to value a risk by (2.1) we have to first specify investor's utility  $u$ , also in the theory of f-expectations in order to value a risk according to Definition 2.1 we have to first specify the generator. The generator  $f$  has an appealing and intuitive interpretation in the terms of the infinitesimal risk measure over the interval  $[t, t + dt]$ , see [1],

$$\mathbb{E}[dY(t)|\mathcal{F}_t] = -f(t, Y(t), Z(t))dt.$$

as it gives the amount of the change in the price over an infinitesimal time period. The process  $Z$  gives the local volatility of this change

$$\mathbb{V}[dY(t)] = |Z(t)|^2 dt.$$

Notice also that by defining the generator as a suitable function, as in points 5-6, we can obtain coherent and convex risk measures. Finally, we would like to refer the reader to [12] where a procedure is given how to estimate  $f$  based observable market

prices.

Let us give examples of pricing mechanisms, already applied in the literature, which can be recovered from BSDEs. The case of the generator

$$f(t, y, z) = \beta(t)z$$

leads to the solution of the corresponding BSDE (2.3) in the form of the expected value under the changed measure

$$Y(t) = \mathbb{E}^{\mathbb{Q}}[\xi | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

where  $\mathbb{Q}$  is an equivalent probability measure given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = e^{\int_0^T \beta(s) dW(s) - \frac{1}{2} \int_0^T \beta^2(s) ds}.$$

If  $\beta = 0$  then  $Y$  reduces to the classical conditional expected value. If we define

$$f(t, y, z) = \gamma(t)|z|$$

then the equation (2.3) yields the robust expected value, defined for  $0 \leq t \leq T$  as

$$\begin{cases} Y(t) = \text{ess inf}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi | \mathcal{F}_t], & \text{if } \gamma(t) < 0 \\ Y(t) = \text{ess sup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi | \mathcal{F}_t], & \text{if } \gamma(t) > 0 \end{cases}$$

where  $\mathcal{M}$  is the set of equivalent probability measures

$$\mathcal{M} = \left\{ \mathbb{Q} : \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = e^{\int_0^T \beta(s) dW(s) - \frac{1}{2} \int_0^T \beta^2(s) ds}, |\beta(t)| \leq |\gamma(t)| \right\}.$$

Such pricing mechanism is especially important in the case when the dynamics of a risk is not known with certainty. Finally, by considering the quadratic generator

$$f(t, y, z) = z^2$$

we recover the exponential premium principle

$$e(t) = \alpha \log \mathbb{E}[e^{\frac{1}{\alpha} \xi} | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Notice that in the last case the generator is not Lipschitz continuous. The existence of a solution of the corresponding BSDE is proved in [10].

In many applications the generator does not have to be specified a priori in a subjective way but it is derived as the result of solving some optimization problem. It means that if a pricing mechanism exists in a market, then we can recover it, by finding the generator in an objective way, and use it to price new instruments. As an illustrative example we consider a pricing and hedging problem in Black-Scholes financial market, firstly without insurance liabilities and secondly with insurance

liabilities.

The Black-Scholes financial market consists of a risk-free bank account

$$\frac{dB(t)}{B(t)} = rdt, \quad (2.4)$$

with a risk-free return  $r$  and a risky asset which price's dynamics is described the stochastic differential equation driven by a Brownian motion  $W$

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad (2.5)$$

where  $\mu$  denotes a return on the risky asset and  $\sigma$  denotes a volatility. An investor issues a liability  $\xi$  (an obligation to cover  $\xi$ ) and wants to sell it a fair price and find a hedging strategy. We first assume that the randomness of  $\xi$  is only due to the randomness of the Brownian motion  $W$ . Let  $(Y(t), 0 \leq t \leq T)$  denotes the value of the portfolio which aims at replicating the pay-off at the terminal times  $T$ , and  $(\pi(t), 0 \leq t \leq T)$  denotes the amount invested in the risky asset  $S$ . The dynamics of the portfolio  $Y$  is given by

$$\begin{aligned} dY(t) &= \pi(t) \frac{dS(t)}{S(t)} + (1 - \pi(t)) \frac{dB(t)}{B(t)} \\ &= \pi(t) (\mu dt + \sigma dW(t)) + (Y(t) - \pi(t)) r dt, \end{aligned} \quad (2.6)$$

with the terminal condition

$$Y(T) = \xi \quad (2.7)$$

which reflects the requirement that an investor wants to construct a replicating portfolio in the sense that the terminal value of the portfolio coincides with the value of the issued liability. By defining

$$Z(t) = \pi(t)\sigma,$$

we can see that the equations (2.6) and (2.7) forms a backward stochastic differential equation with the linear generator of the form

$$f(t, y, z) = -z \frac{\mu - r}{\sigma} - yr$$

Such types of BSDEs can be solved easily. Following [6] we obtain the market value of the portfolio

$$Y(t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}\xi | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

where  $\mathbb{Q}$  is the equivalent probability measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = e^{-\frac{1}{2} \left( \frac{\mu-r}{\sigma} \right)^2 T - \frac{\mu-r}{\sigma} W(T)}, \quad (2.8)$$

and the replicating strategy

$$\pi(t) = \frac{\hat{Z}(t)e^{rt}}{\sigma}, \quad 0 \leq t \leq T,$$

is derived from the martingale representation theorem

$$e^{-rT}\xi = \mathbb{E}^{\mathbb{Q}}[e^{-rT}\xi] + \int_0^T \hat{Z}(s)dW^{\mathbb{Q}}(s).$$

This is well-known result in the Black-Scholes financial model which we have obtained by means of BSDEs.

Assume now that the randomness of  $\xi$  is not only due to the financial risk of  $S$  but also due to an insurance risk. This is the case for unit-linked insurance products. Let  $N := (N(t), 0 \leq t \leq T)$  denote the process which counts the number of survivors in a portfolio of insured lives. We assume that the lifetimes of the insureds are independent from each other and from the financial market, and that they are identically exponentially distributed with intensity  $\lambda := (\lambda(t), 0 \leq t \leq T)$ . Each policyholder is entitled to three types of payments  $g, h, b$ . Firstly, there is an annuity payable continuously, at a rate  $g$ , as long as an insured person is alive. Secondly, there is a benefit in the amount of  $h$  payable at the time of death of an insured person. Thirdly, there is a survival claim in the amount of  $b$  payable provided that an insured person is still alive at the terminal time  $T$ . The liabilities' pay-offs are linked to the value of the risky asset  $S$ . Following the previous example we would like to define the market value of the insurance liabilities as  $Y$  and the replicating strategy as  $\pi$ , where  $(Y, \pi)$  solves the backward stochastic differential equation

$$\begin{aligned} dY(t) &= \pi(t)(\mu dt + \sigma dW(t)) + (Y(t) - \pi(t))r dt \\ &\quad - N(t-)g(t, S(t))dt + h(t, S(t))dN(t), \\ Y(T) &= N(T)b(T, S(T)). \end{aligned} \tag{2.9}$$

If the mortality risk is well diversified in the portfolio, we can replace a random evolution of  $N$  by the deterministic expectations  $M$

$$\frac{dM(t)}{dt} = -M(t)\lambda(t), \quad M(0) = N(0),$$

and consider

$$\begin{aligned} dY(t) &= \pi(t)(\mu dt + \sigma dW(t)) + (Y(t) - \pi(t))r dt \\ &\quad - M(t)g(t, S(t))dt - h(t, S(t))M(t)\lambda(t)dt, \\ Y(T) &= M(T)b(T, S(T)). \end{aligned} \tag{2.10}$$

In life insurance, pricing based on deterministic projections concerning mortality are standard. However we point out that it is a strong assumption which turns an

incomplete combined financial and insurance market into a complete pure financial market. The equations (2.10) constitute a backward stochastic differential equation with a linear generator which, analogously as (2.6)-(2.7), can be solved explicitly. The market value of the insurance liabilities is defined as

$$Y(t) = M(T)u^b(t, T, S(t)) + \int_t^T M(s)u^g(t, s, S(t))ds + \int_t^T M(s)\lambda(s)u^h(t, s, S(t))ds,$$

and the replicating strategy is given by

$$\begin{aligned} \pi(t) = & S(t) \left( M(T)u_s^b(t, T, S(t)) \right. \\ & \left. + \int_t^T M(s)u_s^g(t, s, S(t))ds + \int_t^T M(s)\lambda(s)u_s^h(t, s, S(t))ds \right), \end{aligned}$$

where we define the market prices of the financial claims

$$\begin{aligned} u^b(t, T, S(t)) &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}b(T, S(T))|\mathcal{F}_t], \quad 0 \leq t \leq T, \\ u^g(t, s, S(t)) &= \mathbb{E}^{\mathbb{Q}}[e^{-r(s-t)}g(s, S(s))|\mathcal{F}_t], \quad 0 \leq t \leq s \leq T, \\ u^h(t, s, S(t)) &= \mathbb{E}^{\mathbb{Q}}[e^{-r(s-t)}h(s, S(s))|\mathcal{F}_t], \quad 0 \leq t \leq s \leq T, \end{aligned}$$

under the measure  $\mathbb{Q}$  defined in (2.8), and the subscript  $s$ , in the replicating strategy, denotes the derivative with respect to the third variable of the corresponding function  $u$ . Notice that the above solution is just the price of the pure financial claims from the Black-Scholes models and its delta-hedging replicating strategy, both multiplied by the expected number of the claims to be paid at a given moment.

When the mortality risk cannot be diversified, we cannot consider (2.10). The equation (2.9) does not have a solution and the liabilities cannot be hedge perfectly. We come back to this problem in the next section.

Even in the case when the mortality risk cannot be fully diversified it is possible to perfectly hedge the insurance liabilities. However, an additional instrument has to be traded in the market. Let us consider a simplified example where there is only one insured person and no annuity is paid  $g = 0$ . Assume that in the financial market not only the bank account  $B$  and the stock  $S$  are available, but also a mortality bond is traded which pays 1 if a person survives and 0 otherwise. The mortality bond can be valued in the market-consistent way as

$$V(t) = \mathbb{E}^{\hat{\mathbb{Q}}}[e^{-r(T-t)}\mathbf{1}\{\tau > T\}|\mathcal{F}_t], \quad 0 \leq t \leq T,$$

where the equivalent martingale probability measure is given by the stochastic exponent

$$\begin{aligned} \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} &= \Gamma(T), \\ \frac{d\Gamma(t)}{\Gamma(t-)} &= -\frac{\mu - r}{\sigma}dW(t) + \psi(t)d\tilde{N}(t), \end{aligned} \tag{2.11}$$

and we define

$$N(t) = \mathbf{1}\{\tau > t\}, \quad 0 \leq t \leq T,$$

together with

$$d\tilde{N}(t) = dN(t) - N(t-)\lambda(t)dt, \quad 0 \leq t \leq T.$$

The process  $\psi(t)$  appearing in the Radon-Nikodym derivative (2.11) is an analog of  $\frac{\mu-r}{\sigma}$  and represents the risk premium for investing in the asset  $V$ . By Ito formula we obtain the dynamics of  $V$  in the following form

$$\frac{dV(t)}{V(t-)} = (r + N(t-)\psi(t)\lambda)dt - d\tilde{N}(t). \quad (2.12)$$

We can now construct the replicating portfolio consisting of the three instruments (2.4), (2.5) and (2.12)

$$\begin{aligned} dY(t) &= \pi(t)\frac{dS(t)}{S(t)} + \eta(t)\frac{dV(t)}{V(t-)} + (Y(t-) - \pi(t) - \eta(t))\frac{dB(t)}{B(t)}, \\ Y(\tau \wedge T) &= b(S(T))\mathbf{1}\{\tau > T\} + h(S(\tau))\mathbf{1}\{\tau \leq T\}. \end{aligned} \quad (2.13)$$

such that its value at the random time, either at the time of death of the insured or at the terminal time, coincides with the promised pay-off. The equation (2.13) is of the form

$$\begin{aligned} dY(t) &= -f(t, Y(t), Z(t), U(t))dt + Z(t)dW(t) + U(t)d\tilde{N}(t), \\ Y(\tau \wedge T) &= \xi, \end{aligned}$$

with a linear generator  $f$ . Based on [2] we can conclude that there exists a unique triplet  $(Y, Z, U)$  which solves the BSDE with the uncertain time horizon. The process  $Y$  gives the market value of the liabilities,  $Z$  defines the amount to be invested in the risky asset and  $U$  determines the amount to be invested in the mortality bond.

### 3 Optimization problems and quadratic hedging in incomplete market

The second field of applications of backward stochastic differential equations deals with optimization. A classical way to handle stochastic control problems is to solve the corresponding Hamilton-Jacobi-Bellman equation (HJB). In some cases this approach is sufficient and allows to derive optimal strategies. However, backward stochastic differential equations provide more powerful tool. First of all and most importantly, BSDEs allow to consider non-Markov dynamics of the underlying processes. We recall that Hamilton-Jacobi-Bellman equations are partial differential

equations derived from dynamic programming principle and the Markov assumption, hence exclude a priori any non-Markov behavior. Secondly, BSDEs seem to handle systems with multiple state variables in a more efficient way compared to HJB. Finally, we know that the optimal value function of a considered optimization problem has to be sufficiently smooth in order to satisfy HJB equation in the classical sense. This requires verification which might be challenging and sometimes leads to a conclusion that a classical solution of the underlying HJB does not exist. We may then consider the concept of viscosity solutions but this becomes more involved. BSDEs, contrary to HJB, only have to fulfill some (easier in verification) square integrability conditions. However, BSDEs have also a drawback compared to HJB, as it might happen that for the same optimization problem the corresponding BSDE would not have a solution, whereas HJB would have.

In the previous section we considered the financial market (2.4)-(2.5) and an insurer who issued a unit-linked product with benefits  $g, h, b$ . We have concluded that when the mortality risk cannot be fully diversified we cannot perfectly replicate the insurance liabilities. This is due to the incompleteness of the combined financial and insurance market and/or due to unsolvability of the BSDE (2.9). If the perfect replication is not possible, we could apply some criterion of risk minimization to arrive at a strategy which would cover the claims in the best possible way. A common approach would suggest minimizing the square error

$$\inf_{\pi} \mathbb{E}(Y^{\pi}(T) - L)^2]. \quad (3.1)$$

where  $L$  is the target which an insurance company would like to reach after fulfilling all its obligations. The criterion (3.1) is called a quadratic hedging.

The optimization problem (3.1) can be solved by applying dynamic programming principle and deriving Hamilton-Jacobi-Bellman equation. In this case the optimal value function is a function of 4 variable: time  $t$ , the wealth  $Y$ , the value of the risky asset  $S$  and the number of survivors  $N$ . The smoothness of this optimal value function might be a problem. In the sequel we show how to solve (3.1) by means of BSDEs.

The solution we provide is based on the result from [3], where a more complicated and general optimization problem is investigated. We would like to point out once again that applying backward stochastic differential equations requires mathematical knowledge and rigorous considerations. We only present here a brief conclusion.

The optimal investment strategy which minimizes the square error is of the form

$$\pi(t) = -\frac{\mu - r}{\sigma^2} Y^{\pi}(t-) - \frac{p(t-)(\mu - r) + \beta(t)\sigma}{2\tilde{p}(t)\sigma^2}, \quad (3.2)$$

where  $Y^\pi$  is the value of the portfolio under the strategy  $\pi$  and  $(p, \beta, \eta)$  is a unique triplet solving

$$\begin{aligned} dp(t) &= -\left(p(t-)r - 2\tilde{p}(t)(g(S(t))N(t-) + h(S(t))N(t-)\lambda(t))\right. \\ &\quad \left. - \frac{\mu - r}{\sigma^2}(p(t-)(\mu - r) + \beta(t)\sigma)\right)dt + \beta(t)dW(t) + \eta(t)d\tilde{N}(t), \\ p(T) &= -2b(S(T))N(T) - 2L, \end{aligned} \quad (3.3)$$

and  $\tilde{p}$  solves

$$\begin{aligned} \frac{d\tilde{p}(t)}{dt} &= -\tilde{p}(t)\left(2r - \frac{(\mu - r)^2}{\sigma^2}\right), \\ \tilde{p}(T) &= 1, \end{aligned} \quad (3.4)$$

Next we define the equivalent probability measure by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - \frac{\mu-r}{\sigma} W(T)}. \quad (3.5)$$

Under the measure  $\mathbb{Q}$  the process

$$W^{\mathbb{Q}}(t) = W(t) + \frac{\mu - r}{\sigma}t, \quad 0 \leq t \leq T,$$

is a Brownian motion. Let us define  $\bar{p}(t) = p(t)e^{\delta t}$ ,  $\bar{\beta}(t) = \beta(t)e^{\delta t}$ ,  $\bar{\eta}(t) = \eta(t)e^{\delta t}$  with

$$\delta = r - \frac{(\mu - r)^2}{\sigma^2}.$$

The backward stochastic differential equation (3.3) can be rewritten in the equivalent form under the measure  $\mathbb{Q}$

$$\begin{aligned} d\bar{p}(t) &= -\left(-2e^{\delta t}\tilde{p}(t)(g(S(t))N(t-) + h(S(t))N(t-)\lambda(t))\right)dt \\ &\quad + \bar{\beta}(t)dW^{\mathbb{Q}}(t) + \bar{\eta}(t)d\tilde{N}(t), \\ \bar{p}(T) &= -2(b(S(T))N(T) + L)e^{\delta T}. \end{aligned} \quad (3.6)$$

Again, this is a BSDE with a linear generator which is easy to solve. We can state the solution of (3.3), in particular the pair  $(p, \beta)$  which is required to apply the optimal investment strategy (3.2). We have

$$\begin{aligned} p(t) &= -2e^{(\delta+r)(T-t)}(L + u^b(t, T, S(t))N(t-)w(t, T)) \\ &\quad - 2N(t-) \int_t^T e^{(\delta+r)(s-t)} \tilde{p}(s) \left( u^g(t, s, S(s))w(t, s) + u^h(t, s, S(s))\bar{w}(t, s) \right) ds \Big|_{\mathcal{F}_t} \end{aligned}$$

and

$$\begin{aligned} \beta(t) &= -2e^{(\delta+r)(T-t)}\tilde{p}(t)N(t-)S(t)\sigma w(t, T)u_s(t, T, S(t)) \\ &\quad - 2N(t-)S(t)\sigma \\ &\quad \times \int_t^T e^{(\delta+r)(s-t)}\tilde{p}(s) \left( u_s(t, s, S(s))w(t, s) + u_s(t, s, S(s))\bar{w}(t, s) \right) ds \end{aligned}$$

where

$$\begin{aligned}w(t, s) &= e^{-\int_t^s \lambda(u) du}, \quad 0 \leq t \leq s \leq T, \\ \bar{w}(t, s) &= e^{-\int_t^s \lambda(u) du} \lambda(s) \quad 0 \leq t \leq s \leq T.\end{aligned}$$

We would like to remark that the derived strategy, without difficulties, can be applied in practice.

The interested reader may consult the paper [7] where optimization problems with exponential and power utility functions are investigated.

## 4 Conclusion

In this paper we investigated backward stochastic differential equations and gave their applications to insurance and financial problems. We believe that backward stochastic differential equations, despite they are mathematically challenging, can be applied to real life problems.

There are many possible extensions of BSDEs. One recent extension, considered in [5], deals with time-delayed generators which may depend in path-wise sense on  $(Y, Z)$ . In the forthcoming paper we will present novel applications of BSDEs with time-delayed generators to insurance and finance, see [4].

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