Practical and theoretical aspects of market-consistent valuation and hedging of insurance liabilities

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Abstract

In this paper we deal with market-consistent valuation and hedging of insurance cash flows. We start with recalling traditional actuarial and financial pricing principles and we show how to integrate them into one arbitrage-free principle which leads to market-consistent valuation of the cash flows. Integrated actuarial and financial valuation is justified by referring to Solvency II Directive and discussing its key points related to market-consistent valuation. As an arbitrage-free pricing principle requires specification of an equivalent martingale measure, we characterize all equivalent martingale measures in a very general combined insurance and financial model. This full characterization allows us to price all claims contingent on the financial and insurance risks. We also deal with static and dynamic hedging of insurance liabilities in our general model. We derive an investment portfolio consisting of a bond, a stock and a mortality bond which can be used by a life insurance company to hedge its payment process contingent on the financial and insurance risk. The goal is to unify practical and theoretical aspects of market-consistent valuation and hedging and to state general results relevant to insurance applications

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1   Introduction

For many years insurance and finance were different fields and practitioners and academics did not exchange their views. Insurance products were relatively simple and easy to understand. Financial markets were stable with a limited number of financial instruments and restrictions on capital movements. Pricing of life insurance liabilities involved only deterministic calculations. Life actuaries assumed that the mortality risk could be diversifiable and a portfolio run-off could be replaced by a deterministic scheme. Insurers invested in bonds and real estate which guaranteed a stable growth. The financial risk was negligible and the cash flows were discounted with a low technical rate which could be easily earned in the financial market. The situation changed with deregularization and development of the financial markets. In order to attract investors to insurance products and create a real alternative to financial products, fixed payments under traditional life policies were replaced by benefits linked to risky equities or funds. Moreover, guarantees on policy values and options on surrender values were additionally included. The constructions providing an unbounded upside potential and a downward protection indeed made life insurance products attractive but also made the financial risk a significant factor in valuation of insurance liabilities. The investment side of insurance contracts has made financial mathematics much more important in actuarial science. Over the past years life insurance products have become investment products. Understanding financial markets has become crucial when developing and managing modern life insurance products. Financial hedging has become as important as actuarial pricing and reserving.

The shift from traditional life insurance policies to equity-linked or unit-linked policies had serious consequences. It turned out that traditional solvency principles were not able to cope with new risks. We just recall the most famous case of Equitable Life in the UK, a life insurance company which accepted guarantees on annuity conversion rates in the times of high interest rates and was not able to fulfill the guarantees when the interest rates fell and mortality declined. Actuaries in Equitable Life assumed prudent scenarios concerning interest rates and mortality developments and locked in the assumptions over the duration of the contracts. At the time of issuing the contracts the value of the guarantees was zero under the prudent assumptions (of growing financial markets) and it was not taken into account when pricing, reserving and setting capital requirements. However, the prudent and locked-in assumptions turned out to be too optimistic when financial markets collapsed and mortality decreased. The appropriate hedging positions were not taken and the backing assets mismatched the liabilities. As the result, all guarantees were deeply in the money and Equitable Life became insolvent. Such events forced the European Union to develop a regime under which all risks would be priced and capital requirements would be set in an appropriate and safe way. The Solvency II Directive was

initiated in 2001 and it should be implemented in 2012. Under Solvency II insurance liabilities should be valued in a market-consistent way including all guarantes and options. A market-consistent value of an asset is well understood as in most cases this is the price of the asset observed in the market. However, insurance liabilities are almost never traded. A market-consistent value a liability should be understood as the amount for which the liability could be bought or settled between knowledgeable willing parties in an arm's length transaction. The idea of market-consistent valuation is to translate the insurance cash flows into objective and observable prices which are consistent with the information from the financial market. The market-consistent approach should give much more realistic and objective value of the liabilities in contrast to traditional actuarial valuation which simply does not give a true value. Let us recall that traditional actuarial valuation does not deal with embedded options and guarantees and assumes prudent insurance and investment margins. The market-consistent valuation and calculations of the required solvency capital based on the market-consistent balance sheet subject to stress tests will force insurers to set appropriate hedges and hold financial instruments which match the liabilities as close as possible. We remark that in general the concept of market-consistent valuation concerns life and non-life insurance but it is more prominent in life insurance where financial risk is much more important.

The applications of financial methods to price insurance claims started with the work of Brennan and Schwartz (1976) in which the authors noticed for the first time that a minimum return guarantee on a unit-linked policy could be investigated in the framework of the option pricing theory developed by Black and Scholes. The end of 20th century brought a breakthrough in the way insurance contracts were analyzed. First, Bühlmann (1987) in his visionary paper coined the term of an actuary of the third kind who applied in his daily work both actuarial and financial models and techniques. Next, Delbean and Haezendonck (1989) priced a non-life insurance contract by assuming a non-arbitrage in the market and used martingale methods from financial mathematics. Finally, Aase and Persson (1994) and Briys and Varenne (1994) showed how to use the Black-Scholes theory to value insurance claims with European guarantees. Since the beginning of 21st century we have been observing development and convergence of insurance and financial mathematics. The concepts of market-consistent valuation and hedging of insurance claims have fully crystalized. Hedging techniques for insurance claims have been adopted from financial mathematics of incomplete markets. We should recall Girard (2000) and Sheldon and Smith (2004) who discuss the views of the insurance market on market-consistent valuation, and Embrechts (2000), Schweizer (2001), Møller (2001) who show that the well-known actuarial and financial principles, which have been applied for many years, coincide with each other under some economic assumptions. The works by Møller (2002)

and Steffensen (2000) have built up significantly the mathematical foundations of market-consistent valuation and hedging of insurance claims. These foundations are collected in the book by Møller and Steffensen (2007) which deals with integrated actuarial and financial valuation and hedging of cash flows by taking into account the specific nature of insurance liabilities. At the time when researchers in applied probability were deriving new results in actuarial and financial mathematics, including sophisticated change of measure techniques and hedging methods, the insurance market in the European Union developed Solvency II Directive together with practical rules for market-consistent valuation and replicating portfolios and agreed that market-consistent valuation and hedging were the cornerstone of the modern risk management, see European Commision QIS5 (2010). Last but not least, we recall the book by Wüthrich et al. (2008) in which the main principles of market-consistent valuation and hedging are presented with a mathematical background and a reference to Swiss Solvency Test and Solvency II. Clearly, there are much more papers which have contributed to the subject over the past years but, in the author's opinion, those cited seem to be the most relevant.

In this paper we comment on practical issues related to market-consistent valuation and hedging of insurance liabilities (especially life insurance liabilities) and we deal with advanced mathematical techniques which could be applied to valuation and hedging. The goal is to unify practical and theoretical aspects of market-consistent valuation and hedging and to state general results relevant to insurance applications. We present a motivation for applying market-consistent valuation and hedging from the point of view of the insurance markets and from the point of view of actuarial and financial mathematics, we discuss practical examples and give the corresponding theoretical results, we do simple actuarial calculations and technical calculations based on stochastic processes. We hope that our paper which starts with a basic motivation, goes through practical applications and ends up with theoretical results will be useful to practitioners and theoreticians, actuaries, risk managers and researchers in applied probability. We believe that the subject we touch in this paper is of great importance especially in the times when Solvency II and Enterprise Risk Management strategies are being implemented.

The first part of the paper deals with a philosophy of market-consistent valuation and hedging and practical applications. In Section 2 we start with recalling traditional actuarial and financial pricing principles and we explain the key difference between them. We conclude Section 2 with giving a mathematical justification for integrated actuarial and financial valuation and we end up with an arbitrage-free pricing rule which leads to market-consistent valuation of the cash flows. In Section 3 we support the concept of integrated insurance and financial valuation by referring to Solvency II Directive and discussing its key points related to market-consistent valuation. The first part should serve

as an economic background on the subject before we move to an advanced stochastic models. We finish Section 3 with giving a practical example of market-consistent valuation of a unit-linked contract and we show how to derive the static hedging strategy and the market-consistent price for such product.

The second part of the paper deals with theoretical aspects of market-consistent valuation and hedging. We use advanced modelling techniques and based on these techniques we derive general results concerning valuation and hedging of insurance liabilities. We justify that the theoretical rules comply with the practical principles. We investigate a combined financial and insurance model which is inspired by Becherer (2006), Dahl and Møller (2006),Dahl et al. (2008), Delong (2010). The financial risk is driven by a Brownian motion, the insurance risk is driven by a step process or a random measure with a stochastic intensity and the claim intensity depends on the Brownian motion driving the financial risk, on the step process driving the insurance risk and on a third background source of risk modelled by an independent Brownian motion. We consider a payment process contingent on the financial and insurance risks. This is very general model which includes all important sources of uncertainty needed to be considered when dealing with pricing and hedging of integrated insurance and financial risks. Such a general, and practically relevant, formulation is exceptional in the literature.

Section 4 concerns pricing of insurance and financial claims. As any arbitrage-free pricing rule requires specification of an equivalent martingale measure, we characterize all equivalent martingale measures in our combined insurance and financial model. Very often when dealing with market-consistent valuation of insurance claims the financial part is only valued and the insurance part is totally neglected. If this is the case then known results from the Black-Scholes model can be used. We show how the financial and insurance parts can be priced together by applying the appropriate measure change technique. This uses Girsanov-Meyer theorem from stochastic calculus. Quite general characterizations of equivalent measures in the context of pricing of insurance (defaultable) claims are obtained in Blanchet-Scalliet et al. (2005), Dahl et al. (2008), Dahl and Møller (2006). However, the measure change techniques from Blanchet-Scalliet et al. (2005), Dahl et al. (2008), Dahl and Møller (2006) do not include the important case of a compound Poisson process. In this paper we deal with dynamics driven by a random measure in the spirit of Becherer (2006) which includes a compound Poisson process and the counting processes from Blanchet-Scalliet et al. (2005), Dahl et al. (2008), Dahl and Møller (2006). We also go beyond the change of measure techniques developed for Lèvy processes, see Chapter 9 in Cont and Tankov (2004). Our direct characterization of equivalent martingale measure allows us to price all claims (including streams of claims) contingent on the financial and insurance risk. Those equivalent martingale measures are defined not only in the classical

market consisting of a bank account and a stock but also in the extended market where a mortality bond can be additionally traded. This extension of trading opportunities is of great importance to life insurers. We comment on the usefulness of a mortality bond in pricing and hedging. The characterization of equivalent martingale measures in our extended financial and insurance market reformulates and generalizes the results from Blanchet-Scalliet et al. (2004) and Blanchet-Scalliet et al. (2008) where a purely financial model with a bank account, a stock, a defaultbale bond is considered and the default of the bond is triggered by a one-jump point process with an intensity depending on the evolution of the financial market. Such financial model is complete in the sense that any claim can be hedged perfectly. The combined financial and insurance model considered in this paper is based on multiple defaults (deaths/surrenders) triggered by a stochastic intensity depending on the evolution of the financial market and a background source of uncertainty and, even after the introduction of a mortality bond, the model is still incomplete as the background uncertainty introduces unhedgeable risk.

Section 5 concerns hedging of insurance and financial claims. The classical approach to finding hedging strategies in continuous-time models is to apply a martingale representation theorem. The martingale approach is well-known in the Black-Scholes model. Our model is more general and an appropriate version of a martingale representation theorem exists in stochastic calculus. By following the martingale approach we show how to derive an investment portfolio consisting of the bond, the stock and the mortality bond which can be used by a life insurance company to hedge its payment process contingent on the financial and insurance risks. We consider the cases of perfect hedging and quadratic hedging under an equivalent martingale measure. The results of Section 5 reformulate and extend, in the similar way as in Section 4, the perfect hedging strategy on the complete financial market from Blanchet-Scalliet et al. (2004) and Blanchet-Scalliet et al. (2008). Our quadratic hedging strategy complements both the risk-minimizing strategy from Dahl et al. (2008) in which a portfolio consisting of a bank account, a bond and a mortality swap (another mortality derivative) is derived, and the quadratic hedging strategy from Delong (2010) in which a portfolio consisting of a bank account and a stock is obtained.

We would like to point out that the reformulations and generalizations made in this paper are especially tailored to insurance applications and they are important as the insurers face claims driven by random measures or multiple defaults (deaths/surrenders), the insurers should be interested in buying mortality bonds, they should model a lapse intensity and a mortality intensity as a stochastic process depending on a financial, insurance and background uncertainty.

In Sections 4-5 we use extensively terminology and methods from advanced stochastic calculus and we refer the reader to Applebaum (2004), He et al. (1992) and Protter (2004)

for the details. Classical results from financial mathematics in the Black-Scholes model can be found for example in Shreve (2004).

2   Actuarial valuation vs. financial valuation

Let ξ denote a claim which is faced by an insurer or a bank and which has to be covered at the terminal time T. We are interested in valuating the pay-off ξ. In this section we comment on the philosophy behind actuarial and financial valuation principles which were developed independently from each other. We conclude that actuarial and financial valuations could be considered in the unified framework of market-consistent valuation.

A price for ξ should give an expectation about the final pay-off related to ξ. We can define

$$Price \; for \; \xi = \mathbb{E}[\xi]. \tag{2.1}$$

The price in the form of the expected value (2.1) appears both in insurance and finance but there are significant differences in the meaning of this expected value.

The price (2.1) is intuitive and has strong theoretical foundations. Recall the law of large number which states that for a sequence of independent and identically distributed random variables $(\xi_n)_{n\in\mathbb{N}}$ which has a finite first moment the following convergence holds

$$\lim_{n\to\infty} \frac{\xi_1 + \xi_2 + ... + \xi_n}{n} = \mathbb{E}[\xi], \quad a.s.. \tag{2.2}$$

The equality (2.2) can be interpreted as the effect of diversification. In the case when a portfolio consists of many independent risks which all have the same characteristics (a common distribution function) then the average (random) claim generated by all risks in the portfolio is close to the (non-random) expected value of the claim. Insurance portfolios have been constructed for years with the aim of polling many independent and similar policies. Diversification of risk is the essence of insurance business. Consider a life insurance where a portfolio consists of non-related individuals who belong to the same population. Under the assumption of diversification, which is likely to hold in this example, an uncertain run-off of the portfolio caused by deaths and surrenders could be replaced by a deterministic scheme of decrements according to a mortality and lapse table. This reasoning stands behind all actuarial valuation principles and has been applied in insurance for years. The price $\mathbb{E}[\xi]$ is the fair price for ξ in the sense that the insurer does not loose and does not earn on the business over a finite time horizon. However, charging the expected value is very risky in the long run. It is well-known in the actuarial risk theory, see for example Chapter 4 in Kaas et al. (2001), that the premium principle (2.1) leads to certain ruin in an infinite time horizon where the ruin is defined as the situation

when the paid claims exceed the collected premiums. This theoretical result justifies a safety loading which is applied above the expected value by the insurer. The safety loading should protect the insurer from unfavorable developments in the portfolio. This safety loading can take different forms. In life insurance actuaries usually increase death probabilities and assume prudent investment returns and price and reserve under these conservative conditions. We point out that prudent actuarial and investment assumptions have been a common practice in insurance for many years. However, the level of the prudence introduces arbitrariness in the valuation. We remark that the pricing principle (2.1) could be applied to the discounted pay-off $e^{-rT}\xi$ with the aim of reflecting the time value of money.

Let us point out a crucial point. Notice that the expected value in the above pricing principle is taken under the objective probability measure $\mathbb{P}$ which describes a dynamics of the liability in the real world. From the probabilistic point of view modelling should start with defining a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω describes the set of possible events or the set of scenarios of randomness, $\mathcal{F}$ is a collection of the subsets (the σ-algebra) of Ω and $\mathbb{P}$ is a measure which assigns probabilities to the events or scenarios from Ω. The measure $\mathbb{P}$ defines the real-world probabilities of the events and scenarios.

Example 2.1. Let us toss a fair coin. The real-word measure $\mathbb{P}$ assigns probabilities $\mathbb{P}(tail) = 1/2$ and $\mathbb{P}(head) = 1/2$. Define the random variable

$$\eta = \begin{cases} 1,1 & head \\ 0,9 & tail \end{cases}, \tag{2.3}$$

Assume that we have to cover the claim $\xi = \mathbf{1}\{\eta > 1\}$ and the rate of compound interest is $0,03$. Let us follow the actuarial valuation. The price for insuring the claim ξ according to the expected value principle (2.1), based on the assumption of diversification (2.2), is $e^{-0,03}0,5 = 0,485$. We have taken into account the time value of money and use the real-world probability $\mathbb{P}$. For safety reasons, the insurer would add a loading and would apply conservative assumptions concerning the probabilities. The insurer could use $\mathbb{P}(head) = 3/5, \mathbb{P}(tail) = 2/5$ and would charge $0,582$. The point is that $3/5$ is more or less arbitrary and contains a so-called implicit margin. In contrast to implicit margins, market-consistent valuation advocates explicit margins related directly to the risk or to the views of the market on the risk. $\square$

Financial valuation is different from actuarial valuation. Notice that the diversification argument (2.2) cannot be applied to a pure financial risk. If we consider a portfolio consisting of put options on an equity then increasing the number of put options does not help in managing the risk because all pay-offs are correlated and the convergence result (2.2) does not hold. This is the key difference between financial and insurance risk.

Financial valuation of risk is not based on diversification, as it is the case for actuarial valuation, but on the concept of non-arbitrage. Theoretical foundations of non-arbitrage and risk-neutral pricing are quite difficult and beyond the scope of this introduction. We present only main points and the reader is referred to Delbean and Schachermayer (1994), Delbean and Schachermayer (1998) for the full exposition.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ consisting of increasing σ-algebras (subsets) of $\mathcal{F}$. Intuitively, $\mathcal{F}_t$ contains information about the evolution of a financial market up to time t. Let $X := (X(t), 0 \leq t \leq T)$ denote an investment portfolio which consists of the assets which can be traded continuously in the financial market over the period $[0, T]$. The investment portfolio is called self-financing if its value at time T results from rebalancing and trading the assets. We say that an arbitrage opportunity exists in the financial market if there exists a self-financing portfolio X such that

$$\mathbb{P}(\forall_{t \in [0,T]} X(t) \geq X(0)) = 1, \quad \mathbb{P}(X(T) > X(0)) > 0, \tag{2.4}$$

which means that an investor can gain without incurring a loss. Clearly, arbitrage opportunities should not arise in liquid and well-developed financial markets. By investing we hope to earn but we also bear the risk of loosing the money. The fundamental theorem in mathematical finance states that a financial market is arbitrage-free if and only if there exists an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted prices of the tradeable assets are $\mathbb{Q}$-martingales. Such measure $\mathbb{Q}$ is called an equivalent martingale measure. We recall that an equivalent probability measure $\mathbb{Q}$ is a probability measure which identifies the same, as under $\mathbb{P}$, sets of measure zero:

$$\mathbb{Q} \text{ is equivalent to } \mathbb{P}, \ \mathbb{Q} \sim \mathbb{P} : \quad \mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0, \quad A \in \mathcal{F},$$

and a discounted price process $S := (S(t), 0 \leq t \leq T)$ is a $\mathbb{Q}$-martingale if

$$\mathbb{E}^{\mathbb{Q}}[|S(t)|] < \infty, \quad \mathbb{E}^{\mathbb{Q}}[e^{-rT} S(T) | \mathcal{F}_t] = e^{-rt} S(t), \quad 0 \leq t \leq T,$$

where e^{-rt} is a discount factor related to a risk-free rate r earned in the bank account. More precisely, there exists No Free Lunch with Vanishing Risk in a financial market driven by locally bounded semimartingales if and only if there exists an equivalent probability measure under which the discounted prices of the tradeable assets are $\mathbb{Q}$-local martingales, see Delbean and Schachermayer (1994). The case of unbounded semimartingales can be handled in the framework of $\mathbb{Q}$-sigma-martingales, see Delbean and Schachermayer (1998). The property of No Free Lunch with Vanishing Risk is slightly stronger than the non-arbitrage condition. The non-arbitrage price for a financial claim ξ can be represented as the expected value under an equivalent martingale measure

$$Price \ for \ \xi = \mathbb{E}^{\mathbb{Q}}[e^{-rT} \xi]. \tag{2.5}$$

Example 2.2. Consider a one period asset price model $S(T) = \eta$ where η is defined as in Example 2.1 in (2.3). We still assume that the risk-free rate is $r = 0,03$ and we consider the same claim $\xi = \mathbf{1}\{\eta > 1\}$. We have found the actuarial price for our claim in Example 2.1 by assuming that diversification holds. Let us now follow the financial valuation and assume that the non-arbitrage holds in the financial market consisting of S and the risk-free bank account. The actuarial valuation uses the probability $\mathbb{P}$ and then applies a loading. The financial valuation uses some probabilities under an equivalent martingale measure $\mathbb{Q}$. To find the probabilities $\mathbb{Q}(\eta = 1, 1) = a$ and $\mathbb{Q}(\eta = 0, 9) = 1 - a$ we have to solve the equality

$$1.1a + 0.9(1 - a) = e^{0,03}, \tag{2.6}$$

which follows from the requirement that $e^{-rt}S(t)$ must be a $\mathbb{Q}$ martingale. Notice that a in (2.6) is defined as the probability under which the expected return from the asset S equals the risk-free return. By solving (2.6) we obtain $a = 0,652$. We can derive the non-arbitrage financial price $0,633$. One can say that the role of an equivalent martingale measure $\mathbb{Q}$ in pricing financial risk is similar to the role of a loading in pricing insurance risk but the meaning is different. Intuitively, an equivalent measure $\mathbb{Q}$ reweights, like a loading, the real-world probabilities $\mathbb{P}$ and gives more weight to unfavorable events. However, the new measure $\mathbb{Q}$ arises as the result of the non-arbitrage in the market and is directly related to the observable prices of the tradeable instruments in contrast to an actuarial loading which is specified in a more arbitrary way. The price of $0,633$ has a clear financial interpretation. Assume that we invest ψ in S and ϕ in the bank account and we want to replicate $\mathbf{1}\{\eta > 1\}$ under all scenarios. We have to choose ψ, ϕ to fulfill

$$
\begin{aligned}
1,1\psi + e^{0,03}\phi &= 1, \\
0,9\psi + e^{0,03}\phi &= 0,
\end{aligned}
$$

and we obtain $\psi = 5$ and $\phi = -4,367$. This gives the initial cost of the replicating portfolio which equals to $0,633$. It is easy to notice that for any claim contingent on η we can find a portfolio which replicates the claim perfectly. $\qquad\square$

The results from Delbean and Schachermayer (1994), Delbean and Schachermayer (1998) are powerful and the fundamental theorem of pricing can be applied not only to purely financial claims but can be extended to all claims including insurance claims. Any actuarial valuation principle or financial valuation principle applied in an arbitrage-free market consisting of insurance and financial claims and tradeable assets has to be represented as the expected value operator under an equivalent martingale measure in the form of (2.5). This is a theoretical argument for the convergence of actuarial and financial pricing principles and for integrated actuarial and financial valuation. Such

integrated actuarial and financial valuation is market-consistent in the sense that the valuation operator is calibrated to the arbitrage-free market and the values of the tradeable assets coincides with the observable prices.

In our Example 2.2 the equivalent martingale measure $\mathbb{Q}$ is unique and all claims can be hedged perfectly. Such markets are called complete. In a complete market any claim ξ is valued uniquely by $\mathbb{E}^{\mathbb{Q}}[e^{-rT}\xi]$ and this price coincides with the cost of setting a hedging portfolio which replicates the pay-off in the sense that $X(T) = \xi, a.s..$. The price $\mathbb{E}^{\mathbb{Q}}[e^{-rT}\xi]$ has nothing to do with diversification but it is the cost of perfect replication with the tradeable instruments. However, there are incomplete financial markets in which not all claims can be hedged perfectly. The price for a claim is not unique and we cannot set up perfectly replicating portfolios. There is no unique equivalent martingale measure $\mathbb{Q}$. Different choices of $\mathbb{Q}$ lead to different reweighting of the real-world probabilities. We end up with infinitely many prices but all these prices are consistent with the assumption of non-arbitrage. In fact the class of $\mathbb{Q}$ is restricted. We comment on this in Section 4. First, we discuss how market-consistent valuation and hedging is understood by the insurance market.

3 Integrated actuarial and financial valuation and hedging under Solvency II

Market-consistent valuation and hedging is the cornerstone of Solvency II Directive. We recall the key Solvency II principles which show that integrated actuarial and financial valuation and hedging have been adopted by the insurance market.

The market-consistent value of an insurance liability is the amount for which it could be sold or settled between knowledgeable willing parties in an arm's length transaction, see V.3 in European Commision QIS5 (2010). The idea of market-consistent valuation is to translate the liabilities into objective and observable prices which are consistent with the information from the financial market. The market-consistent pricing model must be calibrated to the prices of the traded assets in the sense that the model price of an asset traded actively in the financial market must coincide with the price of this asset observed in the market. The calibrated market-consistent pricing model is next applied to all other assets and liabilities to derive their market-consistent prices, see V.10 in European Commision QIS5 (2010). Assuming that the market is arbitrage-free this results in pricing assets and liabilities according to the expected value principle under an equivalent martingale measure. Hence, the fundamental theorem of pricing from Delbean and Schachermayer (1994), Delbean and Schachermayer (1998) have found their application in the insurance market and actuarial and financial pricing principles have converged in

the business practice as well.

The price of a liability (or the provision for a liability) under Solvency II is defined as the sum of two elements:

$$Price = Best\ Estimate\ +\ Risk\ Margin. \tag{3.1}$$

The value of a liability should be calculated in a reliable and objective manner and should take into account all cash flows including the intrinsic and time value of options and guarantees, see TP.2.1-11 and TP.2.70 in European Commision QIS5 (2010). The best estimate is calculated as the expected value of the future cash flows weighted with probabilities and discounted with the risk-free interest rate, see TP.2.1 in European Commision QIS5 (2010). The expected value is based on the current information, updated assumptions and unbiased estimates of the future cash flows. It has been agreed that the risk-free interest rate term structure should be based on the swap rates. The probabilities are calibrated so that the expected values of the cash flows from the tradeable instruments coincide with their observed market values. The best estimate is only sufficient for hedgeable risks. In the case of a non-hedgeable risk the risk margin has to be added to protect the insurer from adverse deviations, see TP.5.2-3 in European Commision QIS5 (2010). It has been agreed that the risk margin should be calculated by applying the cost of capital method and the risk margin equals the amount of the funds (the solvency capital) necessary to support the insurance obligations over their lifetime, see TP.5.9 in European Commision QIS5 (2010).

From the practical point of view a hedgeable risk is a risk which can be eliminated by buying financial instruments or transferring it to a willing counterparty in an arm's length transaction under normal business conditions. A hedgeable risk is usually a risk related to the assets which are traded in a deep and liquid financial market. Examples include: short term equity options, interest rate swaps, a rational lapse behavior (optimal lapsation), actively traded securitised risks. A non-hedgeable risk is a risk that cannot be easily transferred to a third party due to market illiquidity. A non-hedgeable risk is usually a risk related to assets or events which are not traded in the financial market or traded in limited amounts. Examples include: long term equity options or interest rate options, an irrational lapse behavior, mortality risk. If we deal with a hedgeable risk (one year option on a traded stock) then the market-consistent price of such risk coincides with the observed price of the risk in the market. For a hedgeable risk the price must be equal to the cost of setting a replicating portfolio consisting of the tradeable assets with the observed market prices, see TP.4.1 in European Commision QIS5 (2010). In the case of a non-hedgeable risk (one year endowment) the market-consistent price of such risk is the sum of the best estimate of the final pay-off and the risk margin for the non-hedgeable risk, see TP.5.2 in European Commision QIS5 (2010). If we deal with a risk that have a

hedgeable and non-hedgeable component (a unit-linked life insurance with an investment guarantee in the case of death) then to arrive at the market-consistent price we should calculate the market price of the hedgeable component, which is the best estimate of the hedgeable component, the best estimate of the non-hedgeable component and add the risk margin for the non-hedgeable component.

Under market-consistent valuation we should try to decompose the liability arising under an insurance contract into blocks of cash flows which can be related to simple financial assets with observable prices. These building blocks can be used to value and hedge the insurance liability. In some cases this is an easy exercise. A trivial case is considered in Example 2.2. It is worth looking into a more practical example.

Example 3.1 Consider a life insurance portfolio which consists of 100 identical unit-linked policies with duration of 4 years. The equity $S := (S(t), 0 \leq t \leq 4)$ which backs the unit-linked fund F is assumed to follow a geometric Brownian motion with the dynamics

$$\frac{dS(t)}{S(t)} = 0.1dt + 0.1dW(t), \quad S(0) = 1.$$

Each policyholder contributes 100 into the fund $F(0) = 100$. The management fee of $m = 3\%$ is deducted each year from the fund. We have

$$F(t) = F(t-1)\frac{S(t)}{S(t-1)}(1-m), \quad t = 1, 2, 3, 4.$$

If a policyholder survives till the end of the contract then he or she receives the final value of the fund $F(T)$. In the case of death of a policyholder the benefit in the amount of the maximum of the fund value and the initial contribution accumulated with the rate of $k = 2,5\%$ per year, $\max\{F(t), F(0)(1+k)^t\}$, is paid. The administration expenses are $e = 10$ euro per policy. The death benefits and expenses are incurred at the end of the year. The future lifetimes of the policyholders are assumed to be independent and identically distributed with death probabilities $q(t)$. No lapses are assumed. In Table 1 the death rates $q(t)$ and the forward rates $f(t-1, t)$ for the periods $t = 1, 2, 3, 4$ are given. We transfer the insurance liability into simple financial assets.

We easily calculate the best estimate of the number of policies $l(t)$ in force

$$l(t) = l(t-1)(1 - q(t)), \quad t = 1, 2, 3, 4, \quad l(0) = 100.$$

The cash flows related to the expenses are obtained from

$$CFexpenses(t) = l(t-1)e, \quad t = 1, 2, 3, 4.$$

These cash flows can be replicated by bonds with different maturities. The market-consistent values of the bonds can be calculated from the appropriate forward rates. The cash flow related to the survival benefit is given by

$$CFsurvival(4) = l(4)F(4).$$

Table 1: Death rates and forward rates.

Year	Death rate	Forward rate
1	5%	4%
2	6%	4.5%
3	7%	5%
4	8%	5.5%

This cash flow can be replicated by buying the equity S. The market-consistent value of the equity at $t = 0$ is given, $S(0) = 1$. The valuation of the death benefit is more complicated. We separate the cash flows related to the death benefits into the cash flows related the death benefit without the guarantee

$$CFdeath(t) = l(t-1)q(t)F(t), \quad t = 1, 2, 3, 4,$$

which can be replicated by buying the equity S and into the cash flows related to the death benefit guarantee

$$CFguarantee(t) = l(t-1)q(t)\big(F(0)(1+k)^t - F(t)\big)^+ \quad t = 1, 2, 3, 4,$$

which can be replicated by buying put options on the underlying equity S with different maturities and strikes. The market-consistent prices of the put options are calculated with the Black-Scholes formula by applying the discount factors related to the appropriate maturities. All calculations can be found in Table 2 together with the exact numbers of the replicating instruments. Notice that the financial risk arising under the unit-linked policies can be hedged perfectly, assuming that the mortality risk is diversified, by buying the bonds, the equity and the put options on the equity. We remark that we construct the replicating portfolio based on the underlying equity S and not on the unit-linked fund F which is the correct approach. The market-consistent best estimate of the liability (cash out-flows) arising from the unit-linked policies equals to $12234, 72$. By the best estimate we mean the discounted expected value of the cash flows related to the replicating financial instruments or the price of the replicating instruments. The market-consistent price of the liability (cash out-flows) would require calculations of the risk margin for the mortality risk which is not considered here. When setting the provision we should include the market-consistent value of the management fees (cash in-flows) as well.

Notice that the traditional actuarial valuation of our unit-linked policies would require specification of a subjective growth rate of the fund which would lead to subjective values of the put options embedded in the product. Such subjective values of the put options, which might be even zero, would be different from the observable objective prices of the options traded in the market. Moreover, a prudent discount rate would be chosen in the

Table 2: Valuation of the unit-linked product.

Year	Discount factor	No. of policies at the beginning	No. of policies at the end	Guaranteed fund value at the end
1	0,94	100,00	95,00	102,50
2	0,89	95,00	89,30	105,06
3	0,83	89,30	83,05	107,69
4	0,78	83,05	76,41	110,38

Year	Best estimate of of the death benefit (without the guarantee)	Best estimate of the survival benefit	Best estimate of the expenses	Best estimate of the liability (without the guarantee)
1	485, 00		943,40	**12117,35**
2	536,31		841,53	
3	570,51		742,76	
4	588,18	6764,09	645,58	

Year	Option price	Best estimate of the guarantee	Best estimate of the death guarantee
1	0,04	18,57	**117,38**
2	0,05	27,24	
3	0,06	33,93	
4	0,06	37,64	

Year	No. of equities	No. of bonds	No. of put options
1	485,00	1000	485,00
2	536,31	950	536,31
3	570,51	893	570,51
4	7352,27	830,49	588,18

actuarial valuation which would also lead to different prices of the bonds. A mismatch would occur. This shows that market-consistent valuation must be applied to report reasonable and coherent values. □

The above example deals with static hedging for insurance liabilities. In Section 5 we investigate more sophisticated dynamic hedging. Dynamic hedging is sometimes the only possibility to manage the risk due to the lack of options in the market. Moreover, dynamic hedging and rebalancing allow to take into account new information from the market. Before we move to hedging, we first introduce a combined financial and insurance model and discuss pricing in our model.

4 The measure change approach to integrated actuarial and financial valuation

The previous sections give a motivation and justification for pricing under an equivalent martingale measure. The crucial point is to construct a model which would be relevant to insurance applications and to characterize all equivalent martingale measures in this model. We have to go beyond the Black-Scholes financial model and include many different sources of uncertainty in the underlying dynamics.

4.1 The financial and insurance model

Let us consider again, but more formally, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and a finite time horizon $T < \infty$. We assume that $\mathcal{F}$ satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). We first introduce a very general combined financial and insurance model.

The financial market is assumed to consist of two tradeable instruments. The price of a risk-free asset (a bank account) $S_0 := (S_0(t), 0 \leq t \leq T)$ is described by the ordinary differential equation

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = 1, \tag{4.1}$$

and the dynamics of a risky asset price $S := (S(t), 0 \leq t \leq T)$ is given by the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW(t), \quad S(0) = s_0 > 0, \tag{4.2}$$

where $r := (r(t), 0 \leq t \leq T)$ denotes the risk-free rate of interest, $\mu := (\mu(t), 0 \leq t \leq T)$ denotes the expected return on the risky asset, $\sigma := (\sigma(t), 0 \leq t \leq T)$ denotes the price

volatility and $W := (W(t), 0 \leq t \leq T)$ is an $\mathcal{F}$-adapted Brownian motion. We assume that

(A1) the processes r, μ, σ are predictable with respect to the natural filtration $\sigma(W(s), 0 \leq s \leq t)$ generated by W and they satisfy

$$\int_0^T |r(t)|dt < \infty, \quad \int_0^T |\mu(t)|dt < \infty,$$

$$\int_0^T |\sigma(t)|^2 dt < \infty, \quad \sigma(t) > 0, \quad 0 \leq t \leq T,$$

$$\left| \frac{\mu(t) - r(t)}{\sigma(t)} \right| \leq K, \quad 0 \leq t \leq T.$$

If r, μ, σ are constants then the model (4.1)-(4.2) reduces to the classical Black-Scholes financial model. It is possible to introduce a multidimensional Brownian motion or a more general driving noise (a Lévy process) in (4.2). We don't consider these extensions on the asset side. We focus more on the liability side.

We consider a stream of insurance liabilities with possible interactions with the financial market. Let $L := (L(t), 0 \leq t \leq T)$ denote an $\mathcal{F}$-adapted step process. We remark that by a step process we mean a process with càdlàg step sample paths that has at most a finite number of jumps in every finite time interval. In the case of life insurance L can count the number of deaths or lapses, in the case of non-life insurance L can count the number of accidents. With the process L we associate the random measure

$$N(dt, dv) = \sum_{s \in (0,T]} \mathbf{1}_{(s, \triangle L(s))}(dt, dv) \mathbf{1}_{\{\triangle L(s) \neq 0\}}(s), \quad (4.3)$$

with $\triangle L(s) = L(s) - L(s-)$ defined on $\Omega \times \mathcal{B}((0,T]) \times \mathcal{B}(\mathbb{R} - \{0\})$. Random measures are often related to discontinuous processes and for the Borel set A the measure $N((0,t], A)$ counts the number of jumps of L of the given height A in the given period $(0,t]$. We assume that

(A2) the random measure N has a unique compensator ϑ, defined on $\Omega \times \mathcal{B}((0,T]) \times \mathcal{B}(\mathbb{R} - \{0\})$, of the form

$$\vartheta(dt, dv) = Q(t, dv)\eta(t)dt,$$

where $\eta : \Omega \times [0,T] \to [0, \infty)$ is an $\mathcal{F}$-predictable process, $Q(\omega, t, .)$ is a probability measure on $\mathbb{R} - \{0\}$ for a fixed $(\omega, t) \in \Omega \times (0,T]$ and $Q(., ., A) : \Omega \times [0,T] \to [0,1]$ is an $\mathcal{F}$-predictable process for a fixed $A \in \mathcal{B}(\mathbb{R} - \{0\})$. The processes satisfy

$$\int_0^T \int_{\mathbb{R}} Q(t, dv)\eta(t)dt < \infty, \quad \eta(t) \geq 0, \quad 0 \leq t \leq T.$$

We set $N(\{0\}, \mathbb{R} - \{0\}) = N((0, T], \{0\}) = \vartheta((0, T], \{0\}) = 0$. The assumption **(A2)** has an intuitive interpretation: η stands for the intensity of the jumps of the process L and Q gives the distribution of the jump given that the jump occurs. Notice that both η and Q can change over time in a random way.

The simplest examples of ϑ are

$$
\begin{aligned}
\eta(t) &= (n - L(t-))\lambda(t), \quad 0 \le t \le T, \\
Q(t, \{1\}) &= 1, \quad 0 \le t \le T,
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
\eta(t) &= \lambda(t), \quad 0 \le t \le T, \\
Q(t, dv) &= q(dv), \quad 0 \le t \le T.
\end{aligned}
\tag{4.5}
$$

The characteristics (4.4) arise in life insurance if we consider a portfolio consisting of n independent individuals who are all subject to a deterministic mortality intensity $\lambda(t)$ and L counts the number of deaths. The characteristics (4.5) are typical of non-life insurance where we model a claim process by a compound Poisson process with a deterministic claim arrival intensity $\lambda(t)$ and a severity of a single claim distributed with q.

We also consider a background driving process $\Lambda := (\lambda(t), 0 \le t \le T)$ with the dynamics given by the stochastic differential equation

$$
d\lambda(t) = a(t)dt + b(t)dW(t) + c(t)dB(t), \quad \lambda(0) = \lambda_0,
\tag{4.6}
$$

where $B := (B(t), 0 \le t \le T)$ denotes an $\mathcal{F}$-adapted Brownian motion independent of W and we assume that

(A3) the processes a, b, c are $\mathcal{F}$-predictable and they satisfy

$$
\int_0^T |a(t)|dt < \infty, \quad \int_0^T |b(t)|^2 dt < \infty, \quad \int_0^T |c(t)|^2 dt < \infty,
$$

In most applications the process Λ affects the intensity η and the role of Λ is to introduce systematic insurance risk. By systematic insurance risk we mean unpredictable changes in the intensity of insurance claims. Recall that by unsystematic insurance risk we understand randomness of insurance claims. In the case of (4.4) we can assume that the mortality intensity in the population $\lambda(t)$ is a stochastic process and we can deal with longevity risk. In the case of (4.5) we can model a claim process by a compound Cox process and we can take into account fluctuations in the expected number of claims due to seasonal variations or expansions/curtailments of a business. These two cases require the introduction of the second Brownian motion B (usually called a background driving

noise) in contrast to financial applications where a default process depends on the same Brownian motion W which drives the financial market. In some insurance applications both (W, B) are needed and the prime example is modelling of an intensity of surrenders depending on the financial market driven by W and an irrational policyholder behavior driven by B.

The insurer usually faces the stream of liabilities $P := (P(t), 0 \le t \le T)$ described by

$$P(t) = \int_0^t H(s)ds + \int_0^t \int_{\mathbb{R}} G(s, y)N(ds, dy) + F\mathbf{1}_{t=T}. \tag{4.7}$$

The process (4.7) may model liabilities arising from various integrated insurance and financial products, including: unit-linked policies, variable annuities, structured products, etc. The loss process P contains payments H which occur continuously during the term of the contract, it contains claims G which occur at random times and liabilities F which are settled at the end of the contract. Based on the representation (4.3) we can conclude that

$$\int_0^t \int_{\mathbb{R}} G(s, y)N(ds, dy) = \sum_{s \in (0,t]} G(s, \triangle L(s))\mathbf{1}_{\{\triangle L(s) \neq 0\}}(s), \quad 0 \le t \le T, \tag{4.8}$$

and we can realize that the stochastic integral with respect to the random measure N models the claims occurring at the times when the step process L jumps. Notice that over a finite time interval the step process L generates a finite number of jumps and the integral (4.8) is a.s. well-defined. We assume that

(A4) the processes H, G are $\mathcal{F}$-predictable and the random variable F is $\mathcal{F}_T$-measurable and they satisfy

$$\int_0^T |H(s)|ds < \infty, \quad \int_0^T \int_{\mathbb{R}} |G(s, y)|^2 Q(t, dy)\eta(t)dt < \infty, \quad |F| < \infty.$$

Notice that H, G and F can depend (also in a pathwise way) on different sources of the uncertainty captured by the filtration $\mathcal{F}$. The liabilities H, G, F may be contingent on the interest rate, stock price, number of deaths, mortality intensity and they may have hedgeable and non-hedgeable insurance and financial components.

The process P based on the underlying three stochastic processes covers all interesting payment schemes that may occur in insurance, reinsurance and pensions, see also Becherer (2006), Dahl and Møller (2006), Dahl et al. (2008), Delong (2010). For example, liabilities arising under a unit-linked portfolio could be represented as

$$\begin{aligned} P(t) &= \int_0^t (n - L(s-))h(s, S(s))ds + \int_0^t \int_{\mathbb{R}} g(s, S(s-))N(ds, dy) \\ &\quad + (n - L(T))f(T, S(T))\mathbf{1}_{t=T}, \quad 0 \le t \le T, \end{aligned} \tag{4.9}$$

with N having the characteristics (4.4) and where h is a survival benefit, g is a death benefit and f is a terminal endowment. Our simple unit-linked portfolio from Example 3.1 fits into the continuous time model (4.9) by setting $h = 0, g(s, S(s)) = \max\{S(t)e^{-mt}, S(0)e^{ks}\}, f(T, S(T)) = S(T)e^{-mT}$. In our opinion the model presented in this section should be used by the insurers in their valuations.

4.2 Pricing by the expected value under an equivalent martingale measure

We would like to price the stream of liabilities P with the dynamics (4.7). Pricing is the key activity of the insurers so a general theory of market-consistent valuation should be outlined.

We assume that

(A5) the combined insurance and financial market is arbitrage-free.

An arbitrage-free price at the time $0 \leq t \leq T$ must be of the form

$$Price \; for \; P \; at \; time \; t = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-\int_s^T r(u)du} dP(s)|\mathcal{F}_t\Big], \quad 0 \leq t \leq T, \qquad (4.10)$$

with some equivalent martingale measure $\mathbb{Q}$. The price (4.10) coincides with the best estimate under Solvency II. Indeed, the price (4.10) is a probability-weighted, updated with the current information, estimate of P discounted with the risk-free rate. Moreover, the measure $\mathbb{Q}$ is calibrated to the tradeable financial instruments. We show how to characterize the equivalent martingale measures in the model introduced in the previous subsection and in its extension with a mortality bond.

First, we need to have a characterization of martingales. This is the key issue not only in pricing but also in hedging. We assume that under our probability space $(\mathbb{P}, \mathcal{F})$ the weak property of predictable representation holds, see Chapter XIII.2 in He et al. (1992), i.e.

(A6) every $(\mathbb{P}, \mathcal{F})$ local martingale $\mathcal{M}$ has the representation

$$\mathcal{M}(t) = M(0) + \int_0^t \psi(s)dW(s) + \int_0^t \phi(s)dB(s) + \int_0^t \int_{\mathbb{R}} \kappa(s,v)\tilde{N}(ds,dv) \quad 0 \leq t \leq T,$$

with $\mathcal{F}$-predictable processes (ψ, ϕ, κ) integrable, in the sense of Itô calculus, with respect to the Brownian motions (W, B) and the compensated random measure $\tilde{N}(dt, dv) = N(dt, dv) - \vartheta(dt, dv)$.

If $\mathcal{M}$ is square integrable then the representation **(A6)** is unique and the stochastic integrals are square integrable martingales. The assumption **(A6)** is almost always satisfied

in insurance and financial models, see Becherer (2006) for detailed constructions and comments.

Let $\mathbb{Q}, \mathbb{P}$ be two equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$. The Radon-Nikodym theorem yields that there exists a positive martingale $M := (M(t), 0 \le t \le T)$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t} = M(t), \quad 0 \le t \le T, \tag{4.11}$$

see Chapter III.8 in Protter (2004). From the representation (**A6**) and the definition of a stochastic exponent we conclude that the positive martingale M must be of the form

$$\frac{dM(t)}{M(t-)} = \psi(t)dW(t) + \phi(t)dB(t) + \int_{\mathbb{R}} \kappa(t,v)\tilde{N}(dt,dv), \quad M(0) = 1, \tag{4.12}$$

with $\mathcal{F}$-predictable processes $\psi := (\psi(t), \le t \le T), \phi := (\phi(t), 0 \le t \le T), \kappa := (\kappa(t,v), 0 \le t \le T, v \in \mathbb{R})$ satisfying

$$\int_0^T |\psi(t)|^2 dt < \infty, \quad \int_0^T |\phi(t)|^2 dt < \infty,$$
$$\int_0^T \int_{\mathbb{R}} |\kappa(t,v)|^2 Q(t,dv)\eta(t)dt < \infty,$$
$$\kappa(t,v) > -1, \quad 0 \le t \le T, v \in \mathbb{R}.$$

The bound on κ guarantees that M is positive. The process (4.12) is a local martingale. If we choose (ψ, ϕ, κ) such that $\mathbb{E}[M(t)] = 1$ for all $t \in [0, T]$ then M is a true martingale and it defines an equivalent probability measure.

Under the new measure $\mathbb{Q}$ defined by (4.11) with the martingale M driven by (ψ, ϕ, κ) the following processes

$$W^{\mathbb{Q}}(t) = W(t) - \int_0^t \psi(s)ds, \quad 0 \le t \le T,$$
$$B^{\mathbb{Q}}(t) = B(t) - \int_0^t \phi(s)ds, \quad 0 \le t \le T,$$
$$\tilde{N}^{\mathbb{Q}}(t,A) = N(t,A) - \int_0^t \int_{\mathbb{R}} (1 + \kappa(s,v))Q(s,dv)\eta(s)ds, \quad 0 \le t \le T, \tag{4.13}$$

are, respectively, $\mathbb{Q}$-Brownian motions and a $\mathbb{Q}$-compensated random measure. This important result follows from the Girsanov-Meyer theorem, see Theorem III.40 in Protter (2004), and some properties of random measures, see Chapter XI.1 in He et al. (1992). If we change the measure to $\mathbb{Q}$ then the stock price S and the background process Λ have the new drifts

$$\frac{dS(t)}{S(t)} = (\mu(t) + \sigma(t)\psi(t))dt + \sigma(t)dW^{\mathbb{Q}}(t),$$
$$d\lambda(t) = (a(t) + b(t)\psi(t) + c(t)\phi(t))dt + b(t)dW^{\mathbb{Q}}(t) + c(t)dB^{\mathbb{Q}}(t),$$

and the step process L has the new intensity of the jumps and the new jump distribution

$$\eta^{\mathbb{Q}}(t) = \eta(t) \int_{\mathbb{R}} (1 + \kappa(t,v))Q(t,dv), \quad Q^{\mathbb{Q}}(t,dv) = \frac{(1 + \kappa(t,v))Q(t,dv)}{\int_{\mathbb{R}} (1 + \kappa(t,v))Q(t,dv)}.$$

In the terminology of traditional actuarial valuation we would say that the change of measure introduces loadings. Notice that the change of measure does not adjust all key parameters of the underlying dynamics. In particular, the volatilities of S and Λ and the support of the jump distribution of L remain unchanged. Changing too much in the dynamics, introducing too many loadings, would destroy the equivalence assumption. The loadings implied by the equivalent measure $\mathbb{Q}$ are not arbitrary but they are directly related to the financial market as we have to work under a martingale measure.

To construct an equivalent martingale measure $\mathbb{Q}$ we can only consider (ψ, ϕ, κ) in (4.12) such that the discounted price processes of the tradeable assets are $\mathbb{Q}$-(local) martingales. Let us consider the financial market introduced in Subsection 4.1. The only tradeable risky asset is the stock S and the discounted price process of the stock $\hat{S} := (\hat{S}(t), 0 \le t \le T)$ is given by the dynamics

$$d\hat{S}(t) = \hat{S}(t)\big((\mu(t) - r(t))dt + \sigma(t)dW(t)\big).$$

Recall now the relation (4.13) for W. In order to make $\hat{S}$ a (local) martingale under $\mathbb{Q}$ we must choose $\psi(t) = -\frac{\mu(t) - r(t)}{\sigma(t)}$. The choice of ψ is unique and the process $\frac{\mu(t) - r(t)}{\sigma(t)}$ is called the market price of the financial risk or the risk premium required by the investors for taking the financial risk. The process ψ clearly coincides with the risk premium in the Black-Scholes model. The processes (ϕ, κ) are free parameters as there are no more tradeable assets which could be used for recovering the values of (ϕ, κ). There exists no unique equivalent martingale measure and no unique non-arbitrage price of a claim in our model, see also Blanchet-Scalliet et al. (2005), Dahl and Møller (2006). We can interpret (ϕ, κ) as the premiums which are required by the investors for taking the systematic and unsystematic insurance risks arising from L, B. We may simply set $\phi = \kappa = 0$ which would correspond to the assumption that the insurance risks can be diversified which is the traditional assumption in actuarial valuation. If we decide on (ϕ, κ) then any liability or a stream of liabilities in our model P contingent on the insurance and financial risk can be priced with the expected value principle (4.10) under the equivalent martingale measure $\mathbb{Q}$ defined by the martingale (4.12) driven by the processes (ψ, ϕ, κ). The processes (ψ, ϕ, κ) should be directly related to the market and the new parameters of the dynamics arising from the change of measure are regarded as objectively implied by the market. The choice of the loadings (ψ, ϕ, κ), which are used when calculating the expected value, is objective in the sense that the processes are derived from the observable and objective prices in the market. This is the advantage of market-consistent pricing over traditional actuarial

valuation where loadings are set more or less in an arbitrary way.

If there are no more instruments than stocks and bonds then it is hard to calibrate the processes (ϕ, κ) in the market-consistent way. Fortunately to the insurance business, additional instruments called mortality bonds are sometimes available.

4.3 Pricing by the expected value under an equivalent martingale measure in the extended model with a mortality bond

Let us extend the financial market (4.1)-(4.2) and assume that a mortality bond is additionally traded. Mortality bonds are gaining popularity as financial instruments for life insurers and some significant issues took place in the past, see for example Blake et al. (2008) and references therein. Many experts believe that mortality bonds would become important hedging instruments in the near future. We would like to incorporate these instruments into our analysis and show their usefulness in pricing and hedging. We derive the dynamics of a mortality bond and we show how to recover the processes (ϕ, κ) needed for market-consistent valuation.

Assume that the processes (ϕ, κ) are decided by the market. The pay-off from a mortality bond is related to the number of survivors in a given population

$$\xi = n - L(T),$$

where L counts the number of deaths and n is the initial number of lives in the reference population. We assume that

(A7) the processes a, b, c are $\mathcal{F}$-predictable with respect to the natural filtration $\sigma(W(s), B(s), 0 \leq s \leq t)$,

(A8) the future lifetimes $(\tau_i)_{i=1,\dots,n}$ in the population are, conditionally, independent and identically distributed with

$$\mathbb{P}(\tau > t | \sigma(W(s), B(s), 0 \leq s \leq t)) = e^{-\int_0^t \lambda(s)ds}.$$

These are classical assumptions in life insurance under longevity risk and in credit risk models, see Dahl et al. (2008), Dahl and Møller (2006), Blanchet-Scalliet et al. (2004), Blanchet-Scalliet et al. (2008). A mortality bond is closely related to a defaultable bond, see Blanchet-Scalliet et al. (2004), Blanchet-Scalliet et al. (2008). In fact, from the point of view of mathematical finance a mortality bond is an extension of a defaultable bond and a defaultable bond arises if $n = 1$ and there is only one driving noise W in the intensity process Λ $(c = 0)$. We remark that a mortality swap, considered in Dahl et al.

(2008), is the second important mortality derivative next to a mortality bond.

Given (ψ, ϕ, κ) the arbitrage-free price of the mortality bond is given by

$$K(t) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds}\big(n - L(T)\big)\big|\mathcal{F}_t\Big], \quad 0 \le t \le T. \tag{4.14}$$

We only consider changes of measure under the condition that

(A9) the processes ϕ, κ are $\mathcal{F}$-predictable with respect to $\sigma(W(s), B(s), 0 \le s \le t)$.

This is a reasonable assumption which implies that the conditional distribution of the future lifetime in the population is still exponential under $\mathbb{Q}$. We can notice that

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds}(n - L(T))\big|\mathcal{F}_t\Big]$$

$$= \mathbb{E}^{\mathbb{Q}}\Big[\sum_{i=1}^n e^{-\int_t^T r(s)ds}\mathbf{1}\{\tau_i > T\}\big|\mathcal{F}_t\Big] = \sum_{i=1}^n \mathbf{1}\{\tau_i > t\}\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds}\mathbf{1}\{\tau_i > T, \tau_i > t\}\big|\mathcal{F}_t\Big]$$

$$= \sum_{i=1}^n \mathbf{1}\{\tau_i > t\}\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds}\mathbb{E}^{\mathbb{Q}}\big[\mathbf{1}\{\tau_i > T, \tau_i > t\}\big|\sigma(W(s), B(s), 0 \le s \le t)\big]\big|\mathcal{F}_t\Big]$$

$$= (n - L(t))\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds}e^{-\int_t^T (1+\kappa(s))\lambda(s)ds}\big|\mathcal{F}_t\Big], \quad 0 \le t \le T, \tag{4.15}$$

where we use the property of conditional expectations, the fact (4.13) that under $\mathbb{Q}$ the intensity of the step process $\mathbf{1}\{t \ge \tau\}$ is changed from $\lambda(t)$ to $(1 + \kappa(t))\lambda(t)$ and the conditional exponential distribution of the future lifetime under $\mathbb{Q}$ which holds due to **(A7)-(A9)**. Let us define $E(t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_0^T r(s)ds}e^{-\int_0^T (1+\kappa(s))\lambda(s)ds}\big|\mathcal{F}_t\big]$, $0 \le t \le T$. Based on the martingale representation **(A6)** we can derive the unique representation of the martingale E in the form

$$E(t) = E(0) + \int_0^t J(s)dW^{\mathbb{Q}}(s) + \int_0^t R(s)dB^{\mathbb{Q}}(s), \quad 0 \le t \le T.$$

Notice that

$$K(t) = (n - L(t))e^{\int_0^t r(s)ds + \int_0^t (1+\kappa(s))\lambda(s)ds}E(t), \quad 0 \le t \le T,$$

and by applying the Itô formula we can obtain the dynamics of the mortality bond

$$\begin{aligned}
dK(t) &= -e^{\int_0^t r(s)ds + \int_0^t (1+\kappa(s))\lambda(s)ds}E(t)dL(t) \\
&\quad + (n - L(t-))e^{\int_0^t r(s)ds + \int_0^t (1+\kappa(s))\lambda(s)ds}E(t)\big(r(t) + (1 + \kappa(t))\lambda(t)\big)dt \\
&\quad + (n - L(t-))e^{\int_0^t r(s)ds + \int_0^t (1+\kappa(s))\lambda(s)ds}\big(J(t)dW^{\mathbb{Q}}(t) + R(t)dB^{\mathbb{Q}}(t)\big) \\
&= K(t-)\Big(\big(r(t) + (1 + \kappa(t))\lambda(t)\big)dt - \frac{\mathbf{1}\{n - L(t-) > 0\}}{n - L(t-)}dL(t) \\
&\quad + \frac{J(t)}{E(t)}dW^{\mathbb{Q}}(t) + \frac{R(t)}{E(t)}dB^{\mathbb{Q}}(t)\Big) \\
&= K(t-)\Big(\big(r(t) + (1 + \kappa(t))\lambda(t) + \frac{J(t)}{E(t)}\frac{\mu(t) - r(t)}{\sigma(t)} - \frac{R(t)}{E(t)}\phi(t)\big)dt \\
&\quad - \frac{\mathbf{1}\{n - L(t-) > 0\}}{n - L(t-)}dL(t) + \frac{J(t)}{E(t)}dW(t) + \frac{R(t)}{E(t)}dB(t)\Big), \tag{4.16}
\end{aligned}$$

where we use the relations (4.13) for (ϕ, κ) and $\psi(t) = -\frac{\mu(t)-r(t)}{\sigma(t)}$. Notice that $K(t) = 0$ if $L(t) = n$.

On the other hand, the dynamics of the non-negative mortality bond K should be of the form

$$dK(t) = K(t-)\Big(\mu^b(t)dt + \sigma_1(t)dL(t) + \sigma_2(t)dW(t) + \sigma_3(t)dB(t)\Big), \qquad (4.17)$$

where the drift μ^b is the real-world return on the mortality bond. By matching the coefficients of (4.16) and (4.17) we obtain that the following relation

$$r(t) + (1 + \kappa(t))\lambda(t) + \sigma_2(t)\frac{\mu(t) - r(t)}{\sigma(t)} - \sigma_3(t)\phi(t) = \mu^b(t), \quad 0 \le t \le T, \qquad (4.18)$$

must hold to guarantee that the discounted price of the mortality bond is a (local) martingale under the measure $\mathbb{Q}$ induced by (ψ, ϕ, κ), compare also with Blanchet-Scalliet et al. (2004), Blanchet-Scalliet et al. (2008). The processes (ϕ, κ) are now related to the price of the traded mortality bond. The equation (4.18) gives an additional and important information on (ϕ, κ) which should be used in market-consistent valuation of the insurance liabilities. If a mortality bond is already traded in the market then valuation of a new insurance risk becomes more objective.

We remark that we follow the intensity approach to modelling of mortality bonds. An alternative approach is based on Heath-Jarrow-Morton framework, see Barbarin (2008).

Even though the change of measure technique is mathematically demanding it is a must if one would like to price complex insurance and financial products in market-consistent framework. The change of measure is also unavoidable when one would like to deal with real-world and risk-neutral, under a martingale measure, simulations as it determines the drifts, intensities and claims distributions which are needed to adjust the simulated paths.

5 The martingale approach to integrated actuarial and financial hedging in the extended model with a mortality bond

In the previous section we have developed a general framework for pricing insurance liabilities. However, in order to manage the financial and insurance risk effectively the insurer should know how to hedge the risk after collecting the premium. A classical approach to finding hedging strategies in continuous-time models is to apply a martingale representation theorem, see for example Chapter 5.3 in Shreve (2004).

We consider a life insurer who faces the payment process P given by the dynamics (4.7). In life insurance the majority of claims could be modelled by a step process with

jump heights of one, $\triangle L(t)\mathbf{1}\{\triangle L(t) \neq 0\} = 1, a.s.$, see Dahl and Møller (2006), Dahl et al. (2008), Delong (2010). Without loss of generality we can assume that the characteristics of L are given by (4.4). The random measure N coincides with L and we have $\tilde{N}(dt) = N(dt) - (n - L(t-))\lambda(t)dt = dL(t) - (n - L(t-))\lambda(t)dt$.

Let $X := (X(t), 0 \leq t \leq T)$ denote an investment portfolio. By $\pi := (\pi(t), 0 \leq t \leq T)$ and $\zeta := (\zeta(t), 0 \leq t \leq T)$ we denote the amount invested in the stock S and in the mortality bond K. Any admissible investment strategy (π, ζ) should be an $\mathcal{F}$-predictable process square integrable in Itô sense. Other strategies could lead to an arbitrage, see Becherer (2006), Delbean and Schachermayer (1994), Delbean and Schachermayer (1998). The portfolio X is traded actively in the sense that the positions in the bank account, the stock and the mortality bond are rebalanced continuously. The dynamics of the investment portfolio is given by the stochastic differential equation

$$
\begin{aligned}
dX(t) =\ & \pi(t)\big(\mu(t)dt + \sigma(t)dW(t)\big) \\
& + \zeta(t)\mathbf{1}\{K(t-) > 0\}\big(\mu^b(t)dt + \sigma_1(t)dL(t) + \sigma_2(t)dW(t) + \sigma_3(t)dB(t)\big) \\
& + \big(X(t) - \pi(t) - \zeta(t)\mathbf{1}\{K(t-) > 0\}\big)r(t)dt \\
& - dP(t), \qquad X(0) = x, && (5.1)
\end{aligned}
$$

where we use the dynamics of the mortality bond (4.17) under the real-world measure $\mathbb{P}$. The value of the portfolio X is decreased by the payments from P. Notice that there is no investment in the mortality bond if it becomes worthless (if the insurer is no longer exposed to the mortality risk).

Assume that in the combined arbitrage-free insurance and financial market the triple (ψ, ϕ, κ) and the equivalent martingale measure $\mathbb{Q}$ is chosen with $\psi(t) = -\frac{\mu(t) - r(t)}{\sigma(t)}$ and (ϕ, κ) fulfilling (4.18) to guarantee that the discounted price processes of (S, K) are $\mathbb{Q}$-(local) martingales. Let $\hat{\zeta}(t) = \zeta(t)\mathbf{1}\{K(t-) > 0\}$. From (5.1) we derive the dynamics

$$
\begin{aligned}
dX(t) =\ & \pi(t)\sigma(t)dW^{\mathbb{Q}}(t) + \hat{\zeta}(t)\sigma_1(t)\tilde{N}^{\mathbb{Q}}(dt) + \hat{\zeta}(t)\sigma_2(t)dW^{\mathbb{Q}}(t) + \hat{\zeta}(t)\sigma_3(t)dB^{\mathbb{Q}}(t) \\
& + \hat{\zeta}(t)\Big(\mu^b(t) - r(t) - \frac{\mu(t) - r(t)}{\sigma(t)}\sigma_2(t) \\
& + (n - L(t-))(1 + \kappa(t))\lambda(t)\sigma_1(t) + \phi(t)\sigma_3(t)\Big)dt + X(t)r(t)dt \\
=\ & \big(\pi(t)\sigma(t) + \hat{\zeta}(t)\sigma_2(t)\big)dW^{\mathbb{Q}}(t) + \hat{\zeta}(t)\sigma_3(t)dB^{\mathbb{Q}}(t) + \hat{\zeta}(t)\sigma_1(t)\tilde{N}^{\mathbb{Q}}(dt) \\
& + X(t)r(t)dt - dP(t), \qquad X(0) = x,
\end{aligned}
$$

where we use the definitions of $\mu^b, \sigma_1, \sigma_2, \sigma_3$ from (4.16) and the compensated random measure $\tilde{N}^{\mathbb{Q}}$. By the change of the variables

$$
\begin{aligned}
Y(t) &= X(t)e^{-\int_0^t r(s)ds}, && 0 \le t \le T, \\
Z_1(t) &= e^{-\int_0^t r(s)ds}\big(\pi(t)\sigma(t) + \hat{\zeta}(t)\sigma_2(t)\big), && 0 \le t \le T, \\
Z_2(t) &= e^{-\int_0^t r(s)ds}\hat{\zeta}(t)\sigma_3(t), && 0 \le t \le T, \\
U(t) &= e^{-\int_0^t r(s)ds}\hat{\zeta}(t)\sigma_1(t), && 0 \le t \le T,
\end{aligned}
\tag{5.2}
$$

we obtain the dynamics of the discountted portfolio

$$
dY(t) = Z_1(t)dW^{\mathbb{Q}}(t) + Z_2(t)dB^{\mathbb{Q}}(t) + U(t)d\tilde{N}^{\mathbb{Q}}(dt) - e^{-\int_0^t r(s)ds}dP(t).
\tag{5.3}
$$

We would like to match the processes (Z_1, Z_2, U) from (5.3) with the processes appearing in the representation of the appropriate martingale. Assume that

(A10) the process P is square integrable under the equivalent martingale measure $\mathbb{Q}$, $\mathbb{E}^{\mathbb{Q}}[|\int_0^T |dP(s)||^2] < \infty$.

There exist unique $\mathcal{F}$-predictable processes $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{U}$ satisfying the following representation

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\int_0^s r(u)du}dP(s)\big|\mathcal{F}_t\Big] &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\int_0^s r(u)du}dP(s)\Big] \\
&+ \int_0^t \mathcal{Z}_1(s)dW^{\mathbb{Q}}(s) + \int_0^t \mathcal{Z}_2(s)dB^{\mathbb{Q}}(s) + \int_0^t \mathcal{U}(s)\tilde{N}^{\mathbb{Q}}(ds), \quad 0 \le t \le T.
\end{aligned}
\tag{5.4}
$$

We can now derive the dynamics of the process $\mathcal{Y}(t) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-\int_0^s r(u)du}dP(s)\big|\mathcal{F}_t\Big]$ in the form

$$
d\mathcal{Y}(t) = \mathcal{Z}_1(t)dW^{\mathbb{Q}}(t) + \mathcal{Z}_2(t)dB^{\mathbb{Q}}(t) + \mathcal{U}(t)\tilde{N}^{\mathbb{Q}}(dt) - e^{-\int_0^t r(s)ds}dP(t).
\tag{5.5}
$$

Clearly, we would like to match Y with $\mathcal{Y}$. However, the existence of a martingale representation is not equivalent to the existence of a perfect hedging strategy. We cannot always recover (π, ζ) from (Z_1, Z_2, U). This is always possible if

(C1) the process P is adapted to the natural filtration $\sigma(W(s), 0 \le s \le t)$,

(C2) the process P is adapted to the natural filtration $\sigma((W(s), L(s))0 \le s \le t)$.

In the first case (C1) we end up with the financial model of Black-Scholes type and we can choose $\mathcal{Z}_2 = \mathcal{U} = Z_2 = U = 0$ in the representations (5.3), (5.5). We have that $\zeta = 0$ and the strategy π is appropriately constructed from $\mathcal{Z}_1$. In the second case (C2) we deal with the combined financial and insurance model driven only by W and L and we can choose $\mathcal{Z}_2 = Z_2 = 0$ as $\sigma_3(t) = 0$. Both processes (π, κ) are appropriately

constructed from $(\mathcal{Z}_1, \mathcal{U})$. The second case arises when there is no systematic mortality risk or when the intensity is stochastic but it depends only on the financial market (this includes the important case of an irrational policyholder lapse behavior), see also Blanchet-Scalliet et al. (2004), Blanchet-Scalliet et al. (2008) and their purely financial model with a defaultable bond. In both cases **(C1)-(C2)** the market is complete in the sense that any square integrable claim can be hedged by applying the appropriate hedging strategy (5.2) derived form the representations (5.3), (5.5). We can easily see the advantage of having a mortality bond as a hedging instruments as it makes perfect replication of insurance liabilities possible. The price of a claim is unique, coincides with the cost of setting the hedging portfolio and equals to the expected value of the discounted pay-off under the unique martingale measure. From (5.5) we can conclude that the arbitrage-free price for P equals

$$
X(0) = \mathbb{E}^{\mathbb{Q}} \Big[\int_0^T e^{-\int_0^s r(u)du} dP(s) \Big],
$$

which agrees with the price we advocate in the previous section.

The representation (5.4) can still be used for hedging in our general incomplete model from Section 3.3 with three sources of uncertainty (W, B, L). Perfect hedging is no longer possible but we can try to hedge our payment process P in the mean square sense. We would like to find an investment strategy (π, ζ) which minimize the following error

$$
\mathbb{E}^{\mathbb{Q}} \Big[\Big| Y(0) + \int_0^T Z_1(t)dW^{\mathbb{Q}}(t) + \int_0^T Z_2(t)dB^{\mathbb{Q}}(t)
$$
$$
+ \int_0^T U(t)d\tilde{N}^{\mathbb{Q}}(dt) - \int_0^T e^{-\int_0^t r(s)ds} dP(t) \Big|^2 \Big]. \tag{5.6}
$$

The criterion (5.6) yields the strategy under which the investment portfolio Y covers the claims P on average (with quantities measured in the discounted values at the inception of the contract). The criterion (5.6) penalizes gains and losses from the insurance policy. Quadratic hedging is very common in financial mathematics and is an extension of Markowitz portfolio optimization problem. Notice that the mean square error in (5.6) is valued under the martingale measure not under the real-world measure. As explained in Chapter 10.4 in Cont and Tankov (2004) this might be the only feasible way to perform quadratic hedging in the case when the drifts in the underlying dynamics are not known or are difficult to estimate.

One can solve the optimization problem (5.6), see also Chapter 10.4 in Cont and Tankov (2004). By using the representation of the discounted payment process (5.4) we

obtain

$$
\mathbb{E}^{\mathbb{Q}}\Big[\Big|Y(0) - \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\int_0^t r(s)ds}dP(t)\Big]
$$
$$
+ \int_0^T Z_1(t)dW^{\mathbb{Q}}(t) + \int_0^T Z_2(t)dB^{\mathbb{Q}}(t) + \int_0^T U(t)d\tilde{N}^{\mathbb{Q}}(dt)
$$
$$
- \int_0^T \mathcal{Z}_1(t)dW^{\mathbb{Q}}(t) - \int_0^T \mathcal{Z}_2(t)dB^{\mathbb{Q}}(t) - \int_0^T \mathcal{U}(t)\tilde{N}^{\mathbb{Q}}(dt)\Big|^2\Big]. \tag{5.7}
$$

By taking the square and applying some properties of stochastic integrals, see Theorem IV.28, Corollary 2 after Theorem II.6.22, Corollary 3 after Theorem II.6.27, Theorem II.6.29 in Protter (2004), and properties of random measures, see Chapter XI.1 in He et al. (1992), we finally arrive at

$$
\Big|Y(0) - \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\int_0^t r(s)ds}dP(t)\Big]\Big|^2
$$
$$
+ \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \big|Z_1(t) - \mathcal{Z}_1(t)\big|^2 dt\Big] + \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \big|Z_2(t) - \mathcal{Z}_2(t)\big|^2 dt\Big]
$$
$$
+ \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \big|U(t) - \mathcal{U}(t)\big|^2(n - N(t-))(1 + \kappa(t))\lambda(t)dt\Big]. \tag{5.8}
$$

By recalling the substitution (5.2) we can conclude that the optimal quadratic hedging strategy is of the form

$$
\pi(t) = \frac{\big(\mathcal{Z}_1(t)e^{\int_0^t r(s)ds} - \hat{\zeta}(t)\sigma_2(t)\big)}{\sigma(t)}, \qquad 0 \le t \le T,
$$
$$
\hat{\zeta}(t) = \frac{e^{\int_0^t r(s)ds}\big(\mathcal{Z}_2(t)\sigma_3(t) + + \mathcal{U}(t)\sigma_1(t)(n - N(t-))(1 + \kappa(t))\lambda(t)\big)}{\sigma_3^2(t) + \sigma_1^2(t)(n - N(t-))(1 + \kappa(t))\lambda(t)}, \quad 0 \le t \le T,
$$

which is the unique minimizer of the quadratic functional (5.8). Under additional assumptions (which we omit) the strategy (π, ζ) is square integrable, hence it is admissible. Notice that (5.8) yields that the price for P should be $X(0) = \mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\int_0^t r(s)ds}dP(t)]$, hence the market-consistency is satisfied under quadratic hedging. The mortality bond still improves the insurer's hedging programme as keeping the mortality bond as the hedging instrument reduces the cost of replication measured by the mean square error (5.6). Without the mortality bond we could not control the two last integrals in (5.8).

By using the approach of this Section we can find a dynamic hedging strategy for the unit-linked product from Example 3.1. If we assume that the mortality risk is diversified then we consider the model under the assumption (C1). We can conclude that if mortality is diversified then the claims are purely financial and the stream of liabilities can be hedged by investing in the bank account and the stock. We have already shown this in Example 3.1. Notice that the replicating dynamic strategy π derived in this section is in terms of the underlying equity S and the bank account B whereas the static replicating

strategy from Example 3.1 is in terms of the bonds, the equity and the put options on the equity. Both strategies clearly coincide and yield the same price as the cost of the replicating portfolio. If the mortality risk is not diversified but there is no systematic mortality risk then we consider the model under the assumption **(C2)**. In this case perfect hedging of the insurance liability is possible if we apply dynamic rebalancing of funds between the bank account, the stock and the mortality bond as explained in this section. Finally, if there exists systematic mortality risk then we can still hedge our payment process in the mean square sense by following the appropriate strategy.

Dynamic hedging requires to derive the martingale representation of the liability P under $\mathbb{Q}$. For many insurance claims this can be found. We refer the reader to Section 6 in Delong (2010) where some examples are given including the martingale representation of the payment process related to the unit-linked policy discussed in Example 3.1.

Finally, we would like to comment on the technical conditions which have been stated throughout the paper. The assumptions made in the paper are needed so that the stochastic integrals exist and the martingale representations hold. Most of the assumptions are straightforward, intuitive and immediate, but they are necessary for the applications of stochastic calculus. The most strict assumptions concern square integrability of the processes and they are unavoidable as otherwise arbitrage strategies could arise, see Becherer (2006), Delbean and Schachermayer (1994) and Delbean and Schachermayer (1998).

6 Conclusion

In this paper we have discussed market-consistent valuation of insurance liabilities and we have dealt with static and dynamic hedging of insurance cash flows. We have introduced and developed the concepts of market-consistent valuation and hedging both from the practical point of view of the insurance market and from the theoretical point of view of mathematical finance. We hope that this work clarifies the key issues related to market-consistent valuation and hedging of insurance liabilities and points out some advanced mathematical methods which should be recognized when dealing with valuation and hedging under Solvency II regime.

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