# Asset allocation, sustainable withdrawal, longevity risk and non-exponential discounting


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**Abstract**

The present paper studies an optimal withdrawal and investment problem for a retiree who is interested in sustaining her retirement consumption over a pre-specified minimum consumption level. Apparently, the withdrawal and investment policy depends substantially on the retiree's health condition and her time preferences (subjective discount factor). We assume that the health of the retiree can worsen or improve in an unpredictable way over her lifetime and model the retiree's mortality intensity with a stochastic process. In order to make the decision about the consumption and investment policy more realistic from the economic point of view, we assume that the retiree applies a non-exponential discount factor (by adding a small amount of hyperbolic discounting to exponential discounting) to value her future income. In other words, we consider an optimization problem by combining four important aspects: asset allocation, sustainable withdrawal, longevity risk and non-exponential discounting. Due to non-exponential discount factor, we have to solve a time-inconsistent optimization problem. We derive an HJB equation which characterizes the equilibrium optimal investment and consumption strategy and solve the HJB equation by applying expansion techniques. We find the first order approximations to the equilibrium optimal consumption and investment strategy.

Keywords: Hyperbolic discounting, time-inconsistent optimization problem, non-local HJB equation, equilibrium strategies, PDE.
JEL:

# 1 Introduction

The primal objective of a retiree is to sustain a living standard in her retirement life comparable to the working life. In order to achieve this goal, the retiree needs to know how to invest the accumulated assets in the financial market and how to consume the stream of retirement income from the accumulated assets. Too little withdrawal of the available funds for consumption might not ensure the desired living standard, while too much withdrawal might lead to exhaustion of the accumulated funds too soon. Apparently, the investment and withdrawal policy depends on diverse factors, among which there are two important factors, i.e. longevity risk and the time preferences of the retiree (i.e. the retiree's subjective discount factor).

In this paper we study an optimal consumption and investment problem for a retiree who first buys a lifetime annuity providing a certain future income and next she manages the available assets by investing them in the financial market with the aim of increasing the future consumption over a pre-specified minimum consumption level. Clearly, the retiree prefers to consume her savings before death but at the same time the funds cannot be exhausted before death. The introduction of a minimum consumption level in the optimization problem means that the retiree cares about sustaining a minimum living standard at her retirement life. Sustainable consumption becomes especially relevant when the retiree lives longer than expected. We assume that the health of the retiree can worsen or improve in an unpredictable way over her lifetime and we model the retiree's mortality intensity with a stochastic process. Consequently, the consumption and investment strategy must be appropriately adapted to the health condition of the retiree. Furthermore, in order to make the decision about the consumption and investment policy more realistic from the economic point of view, we assume that the retiree applies a non-exponential discount factor to value her future income. There is strong evidence that people discount the future income with non-constant rates of time preferences. Experimental studies by psychologists and economists suggest that rates of time preference tend to decline as a function of the horizon over which the utility is discounted. In other words, people's valuation tends to decrease rapidly for short period delays and less rapidly for longer period delays, see Loewenstein and Prelec (1992), Luttmer and Mariotti (2003), Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Ekeland et. al. (2012). An intuitive example from the literature will be that people prefer to get two oranges in 21 days than one orange in 20 days, but also prefer to get one orange now than two oranges tomorrow. Such a feature is called the common difference effect which cannot be described by exponential discounting, but can be described by hyperbolic discounting. In the present paper, we consider discount factors which arise from exponential discount factors perturbed by adding a small amount of hyperbolic discounting. Such discount factors imply declining rates of time preferences.

Since the paper by Merton (1971) the problem of finding optimal consumption and investment policy has become the most studied optimization problem in economy, finance and insurance. The present paper contributes to this stream of literature by combining four important aspects: asset allocation, sustainable withdrawal, longevity risk and non-exponential discounting. The pure impact of stochastic mortality on consumption strategies has already been considered in the literature. Richard (1975) and Pliska and Ye (2007) are the first who extend the Merton problem by considering an agent who can die at some random time. They model the agent's mortality intensity with a deterministic function and find the optimal consumption and investment strategies for power utility and exponential discounting. Stochastic mortality intensity has been recently investigated in Guambe and Kufakunesu (2015) and Shen and Wei (2016), where the authors solve an optimal consumption and investment problem for an agent with no future income for power utility and exponential discounting. The authors assume that the stochastic mortality can be modelled with a continuous stochastic process or a stochastic process with jumps. Huang and Milevsky (2011) concentrate on the effect of longevity risk aversion on consumption and they consider a deterministic force of mortality. Their study suggests that wealth managers should advocate dynamic spending in proportion to survival probabilities, adjusted up for exogenous pension income and adjusted down for longevity risk aversion. Finally, Huang et. al. (2011) investigate a stochastic mortality intensity modelled with a diffusion process and the authors find an optimal consumption (without investment strategy) for an agent with no future income for power utility under exponential discounting. Our paper extends the result from Huang et. al. (2011) and Huang and Milevsky (2011) by taking into account asset allocation, non-exponential preferences, stochastic mortality intensity process, future income and minimum consumption level, and the result from Guambe and Kufakunesu (2015) and Shen and Wei (2016) by considering non-exponential preferences, future income and minimum consumption level. We would like to remark that it is important to include future income in retiree's decision making process since in real life frequently the retiree receives a lifetime annuity during her retirement life.

Also the impact of non-exponential discounting on optimal asset allocation and consumption is not new in the literature. The optimization problem with non-exponential discounting leads to time-inconsistent strategies, which makes the application of Bellman principle of optimality impossible. Björk and Murgoci (2010) develop a theory for solving time-inconsistent optimization problems. They derive an extended version of a classical HJB equation which contains a non-local term, due to which explicit solutions to the HJB equations are unlikely to be found. Similar to our problem, Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Ekeland et. al. (2012) consider an optimal investment and consumption problem for an agent with general time-inconsistent preferences and deterministic future income. The most closely related to our research is the paper by Ekeland et. al. (2012) where the authors assume that the agent is exposed to mortality risk modelled with a deterministic mortality intensity function. The authors derive an HJB equation with a non-local term characterizing the value function of the optimization problem and the optimal strategies. Since the HJB equation cannot be solved explicitly, the authors present a numerical algorithm to solve the HJB equation. As a resolution to dealing with complicating HJB equations with non-local terms which arise under time-inconsistent preferences, Dong and Sircar (2014) suggest to apply an expansion technique for solving non-local HJBs. Dong and Sircar (2014) solve the classical Merton problem for an agent with exponential discounting perturbed with a small amount of hyperbolic discounting and derive explicit formulas for the first order approximations to the optimal consumption and investment strategies. Our paper extends the result from Dong and Sircar (2014) by taking account of mortality, stochastic mortality intensity process, future income and minimum consumption level, and the result from Ekeland et. al. (2012) (and Ekeland and Lazrak (2006), Ekeland and Pirvu (2008)) by introducing stochastic mortality intensity process, minimum consumption level and deriving the optimal result which are applicable.

The remainder of the paper is organized as follows. Section 2 describes the underlying financial market and the stochastic mortality process. Section 3 formulates the optimal asset allocation and withdrawal problem. In the subsequent Section 4, we solve our optimization problem. In Section 5, we provide interpretation of the optimal equilibrium strategies and include a numerical example. All proofs are included in the Appendix.

# 2 Model setup

We deal with a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ and a finite time horizon $T < \infty$. On the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ we define a non-negative random variable $\tau$ and a two-dimensional standard Brownian motion $W = (W_m, W_\lambda) = (W_m(t), W_\lambda(t), 0 \le t \le T)$. The Brownian motions $W_m$ and $W_\lambda$ are independent. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ consists of three subfiltrations. We set $\mathcal{F}_t = \mathcal{F}_t^m \vee \mathcal{F}_t^\tau \vee \mathcal{F}_t^\lambda$ for all $0 \le t \le T$, where $\mathcal{F}_t^m = \sigma(W_m(u), u \in [0, t])$ contains information about the financial market, $\mathcal{F}_t^\tau = \sigma(\mathbf{1}\{\tau \le u\}, u \in [0, t])$ contains information on the survival of the retiree and $\mathcal{F}_t^\lambda = \sigma(W_\lambda(u), u \in [0, t])$ contains information about the mortality intensity of the retiree. We assume that the subfiltrations $\mathcal{F}_t^m$ and $(\mathcal{F}_t^\tau, \mathcal{F}_t^\lambda)$ are independent, i.e. we assume that

(A1) the financial risk is independent of the insurance risk.

As always, the filtration $\mathbb{F}$ is completed with sets of measure zero.

The financial market consists of a risk-free bank account and a risky asset. The value of the risk-free bank account $B = (B(t), 0 \leq t \leq T)$ growths at an exponential rate, i.e. is the value process $B$ satisfies the differential equation

$$\frac{dB(t)}{B(t)} = rdt, \quad 0 \leq t \leq T, \quad B(0) = 1, \tag{2.1}$$

where $r > 0$ denotes the instantaneous rate of interest paid on cash deposited in the bank account. The price of the risky asset $S = (S(t), 0 \leq t \leq T)$ is modelled with a geometric Brownian motion, i.e. the price process $S$ satisfy the stochastic differential equation (SDE)

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_m(t), \quad 0 \leq t \leq T, \quad S(0) = 1, \tag{2.2}$$

where $\mu > r$ denotes the instantaneous expected rate of return on the risky asset and $\sigma > 0$ denotes the volatility of the rate of return.

The future lifetime of the retiree is modelled with the random variable $\tau$ which is assumed to take values in the set $[0, T] \cup \{+\infty\}$. We use the convention that $\tau = +\infty$ if $\tau > T$. The distribution of the future lifetime of the retiree is determined by the conditional survival probability

$$\mathbb{P}(\tau \geq t | \mathcal{F}_t^\lambda) = e^{-\int_0^t \lambda(u) du}, \quad 0 \leq t \leq T, \tag{2.3}$$

where $\lambda$ denotes the mortality intensity of the retiree. Following Huang et. al. (2011), Luciano and Vigna (2008) and Vigna et. al. (2008), we use Brownian motion to model randomness in the future mortality intensity $\lambda$. More specifically, we assume that the mortality intensity of the retiree $\lambda = (\lambda(t), 0 \leq t \leq T)$ evolves randomly in time in accordance with the dynamics

$$d\lambda(t) = p(t, \lambda(t))dt + q(t, \lambda(t))dW_\lambda(t), \quad 0 \leq t \leq T, \quad \lambda(0) = \lambda_0. \tag{2.4}$$

By this way of modelling, we consider the fact that the health of the retiree can get worse or better in an unpredictable way during her lifetime. For example, in some future point of time the retiree can go to a hospital and have an operation. After that operation the retiree's health condition can get worse and the mortality intensity can increase at a faster rate than expected. [1] We assume that the retiree continuously "observes" her changing mortality intensity, i.e. the retiree has perfect knowledge about her current heath condition and the distribution of her future lifetime.

The dynamics (2.4) which defines the evolution of the mortality intensity in time is very general. We require that

(A2) the process $\lambda$ is bounded from below and from above, i.e. the process $\lambda$ takes values in a bounded, open and connected set $\mathcal{K}$,

(A3) the functions $p, q : [0, T] \times \mathcal{K} \mapsto \mathbb{R}$ are uniformly Lipschitz continuous in $(t, \lambda)$.

An obvious lower bound for the mortality intensity $\lambda$ is zero. It is known that under (A2)-(A3) there exists a unique solution to the SDE (2.4), see Theorem 2.9 in Karatzas and Shreve (1988).

Let us introduce an operator associated with the dynamics of the the process $\lambda$.

**Definition 2.1.** *Let $\mathcal{L}_\lambda$ denote a second order differential operator given by*

$$\mathcal{L}_\lambda v(t, \lambda) = \frac{\partial}{\partial \lambda} v(t, \lambda) p(t, \lambda) + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} v(t, \lambda) q^2(t, \lambda). \qquad (2.6)$$

*The operator $\mathcal{L}_\lambda$ is defined for $v \in \mathcal{C}^{1,2}([0, T] \times \mathcal{K})$.* [2]

[1] It might be more reasonable to assume that the health of the retiree changes only at some random points of time due to some random events (such as a hospital visit and operation) and that the retiree can only be aware of significant changes in her health condition which are caused by important (and seldom) events. In such a case the dynamics of the mortality intensity $\lambda$ should be modelled with the SDE

$$d\lambda(t) = p(t, \lambda(t))dt + q(t, \lambda(t))dJ(t), \quad 0 \le t \le T, \quad \lambda(0) = \lambda_0, \qquad (2.5)$$

where $J$ denotes a compound Poisson process. In this paper we solve our optimization problem assuming the dynamics (2.4). However, we would like to point out that the solution to our optimization problem would be the same if we assumed the dynamics (2.5). We comment on this point later during our calculations.

[2] If we considered the dynamics (2.5), then the operator associated with the process $\lambda$ would take

# 3 The optimal asset allocation and withdrawal problem

Think of a retiree who is endowed with a wealth level $\hat{x} > 0$ at her retirement date (in our framework at time $t = 0$). Since nowadays retirees are highly encouraged to invest in private retirement plans, we assume that at time $t = 0$ the retiree spends a part of her initial wealth $\alpha\hat{x}$, $0 \leq \alpha \leq 1$, on a lifetime annuity. From this investment the retiree will receive a lifetime income at the rate $a(\alpha\hat{x})$. More precisely, by spending $\alpha\hat{x}$ on the annuity the retiree buys an annuity which pays $a(\alpha\hat{x})$ for lifetime, but not longer than $T$ years. The annuity income $a(\alpha\hat{x})$ is determined by the annuity provider. The remaining money $(1-\alpha)\hat{x}$ is used by the retiree to set up an investment plan and to increase the consumption which will be guaranteed by the lifetime annuity. Furthermore, we assume that the retiree is interested in maintaining a given living/consumption standard and has some bequest motives. The retiree needs to decide about how to invest and how much to withdraw from the available funds to consume during the retirement life. In our model we consider a fixed time horizon $T$, however, the time horizon $T$ can be chosen so that $Pr(\tau > T)$ is negligible and in fact we can deal with an uncertain time horizon determined by the retiree's death.

Let $\pi(t)$ denote the fraction of wealth that the retiree invests in the risky asset and $(1 - \pi(t))$ the fraction in the risk-free bank account, and $c(t)$ the consumption rate. The wealth process of the retiree $X = (X(t), 0 \leq t \leq T)$ satisfies the SDE

$$\begin{aligned} dX(t) &= \pi(t)X(t)\big(\mu dt + \sigma dW_m(t)\big) + (1-\pi(t))X(t)rdt \\ &\quad -c(t)dt + a(\alpha\hat{x})dt, \quad 0 \leq t \leq T, \\ X(0) &= (1-\alpha)\hat{x}. \end{aligned} \tag{3.1}$$

the form

$$\mathcal{L}_\lambda v(t,\lambda) = \frac{\partial}{\partial\lambda} v(t,\lambda)p(t,\lambda) + \int_{\mathbb{R}} (v(t,\lambda + q(t,\lambda)z) - v(t,\lambda))\zeta\vartheta(dz), \tag{2.7}$$

where $\zeta$ denotes the intensity of the compound Poisson process $J$ and $\vartheta$ denotes the distribution of the jumps.

To sustain a certain living standard, the retiree's consumption $c$ is assumed to be above a minimum consumption rate of $c^*$. Hence, we introduce the decomposition

$$c(t) = c^* + u(t), \quad 0 \leq t \leq T, \tag{3.2}$$

where $u$ denotes the excess of the realized consumption over the minimum consumption $c^*$. In the literature on habit-formation, the constant $c^*$ is interpreted as a habit level, which is usually defined as a function of past consumption rates, see e.g. Munk (2008). Here, we assume that $c^*$ is a positive constant.

We need a condition which guarantees that the minimum consumption rate $c^*$ can be sustained by a retiree who has a capital $\hat{x}$ at her disposal. We assume that

(A4) $\hat{x} \geq \int_0^T c^* e^{-rt} dt = c^* \frac{1-e^{-rT}}{r}$.

If (A4) is satisfied, then the retiree does not have to buy a life annuity, she can deposit the whole initial wealth into the risk-free bank account and she will be able to consume $c^*$. If (A4) is satisfied, then the retiree can alternatively spend the whole initial wealth on the life annuity and the annuity bought will provide a certain income higher than $c^*$. Consequently, if (A4) holds, then any $\alpha$ in the range $[0, 1]$ is admissible and the retiree at the retirement date can consider all range of possible investment policies which allow her to sustain a minimum consumption. The retiree can consider full annuitization, no annuitization, partial annuitization together with dynamic allocation of the available funds into the risk-free bank account and the risky asset.

The controlled wealth process $X^{\pi,u}$ takes the form

$$\begin{aligned} dX^{\pi,u}(t) &= \pi(t) X^{\pi,u}(t)\big(\mu dt + \sigma dW_m(t)\big) + (1 - \pi(t)) X^{\pi,u}(t) r dt \\ &\quad - c^* dt - u(t) dt + a(\alpha \hat{x}) dt, \quad 0 \leq t \leq T, \\ X^{\pi,u}(0) &= (1 - \alpha)\hat{x}. \end{aligned} \tag{3.3}$$

We introduce an operator associated with the dynamics (3.3) and the set of admissible strategies.

**Definition 3.1.** *Let $\mathcal{L}_x$ denote a second order differential operator given by*

$$\begin{aligned}\mathcal{L}_x^{\pi,u}v(t,x) &= \frac{\partial}{\partial x}v(t,x)\big(\pi x(\mu-r)+xr+a(\alpha\hat{x})-c^*-u\big)\\&+\frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x)\pi^2x^2\sigma^2. \qquad (3.4)\end{aligned}$$

*The operator $\mathcal{L}_x^{\pi,u}$ is defined for $v\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$.*

In the sequel the partial derivatives with respect to the state variables (the mortality intensity, the wealth process) and time are denoted by $v_\lambda, v_{\lambda\lambda}, v_x, v_{xx}, v_t$.

**Definition 3.2.** *A strategy $(\pi,u)=(\pi(t),u(t),0\le t\le T)$ is called admissible, $(\pi,u)\in\mathcal{A}$, if it satisfies the following conditions:*

1. *$\pi,u:[0,T]\times\Omega\to\mathbb{R}$ are progressively measurable mappings with respect to filtration $\mathbb{F}$,*
2. *The process $u$ is non-negative, $u(t)\ge 0,\ 0\le t\le T$, and the terminal wealth is non-negative, $X^{\pi,u}(T)\ge 0$,*
3. *$\mathbb{E}\Big[\int_0^T|\pi(t)X^{\pi,u}(t)|^2dt+\int_0^T|u(t)|^2dt\Big]<\infty$,*
4. *The stochastic differential equation (3.3) has a unique solution $X^{\pi,u}$ on $[0,T]$,*
5. *The strategy $(\pi,u)$ is a Markov strategy.*

If $(\pi,u)$ is a Markov strategy, then $\pi(t)=\pi(t,X^{\pi,u}(t),\lambda(t))$ and $u(t)=u(t,X^{\pi,u}(t),\lambda(t))$. If $(\pi,u)\in\mathcal{A}$, then the solution $X^{\pi,u}$ to the SDE (3.3) is a continuous, $\mathbb{F}$-adapted process, see Theorem 2.9 in Karatzas and Shreve (1988). Moreover, from (3.3) we get the representation

$$\begin{aligned}X^{\pi,u}(t) &= (1-\alpha)\hat{x}e^{rt}+\int_0^t\pi(s)X^{\pi,u}(s)e^{r(t-s)}(\mu-r)ds\\&+\int_0^t\pi(s)X^{\pi,u}(s)e^{r(t-s)}\sigma dW_m(s)\\&-\int_0^t e^{r(t-s)}(u(s)+c^*-a(\alpha\hat{x}))ds,\quad 0\le t\le T, \qquad (3.5)\end{aligned}$$

and using Bulkholder inequality and point 3 from Definition 3.2 we can derive the moment estimate

$$\mathbb{E}[\sup_{t\in[0,T]} |X^{\pi,u}(t)|^2] \leq K\Big(1+\mathbb{E}\Big[\int_0^T |\pi(s)X^{\pi,u}(s)|^2 ds + \int_0^T |u(s)|^2 ds\Big]\Big) < \infty. \quad (3.6)$$

Hence, for $(\pi, u) \in \mathcal{A}$ the solution $X^{\pi,u}$ is square integrable.

We assume that the retiree's preference for consumption and bequest is modelled by a power utility function with a relative risk aversion coefficient $1-\gamma$. We assume

(A5) the relative risk aversion coefficient of the power utility $1-\gamma \in (0,1)$.

At the retirement date $t = 0$, the retiree is interested in solving the following optimization problem

$$\sup_{(\pi,u)\in\mathcal{A}} \quad \mathbb{E}\left[\int_0^{\tau\wedge T} \phi(t)u(t)^\gamma dt + q\phi(T)(X^{\pi,u}(T))^\gamma 1_{\{\tau>T\}}\right], \quad (3.7)$$

where the parameter $q > 0$ describes how much the retiree weighs the bequest in the total expected utility and the function $\phi$ specifies subjective discount factors which are applied by the retiree to value future income. We choose the non-exponential discounting function

$$\phi(t) = e^{-\rho t}\frac{1}{(1+\delta t)^\varepsilon}, \quad 0 \leq t \leq T, \quad (3.8)$$

where $\rho \geq 0, \delta \geq 0, \varepsilon \geq 0$. We use the discount factors (3.8) which are exponential discount factors perturbated with hyperbolic discounting. The reason why we choose the discounting function (3.8) is that we want to consider rates of time preference that decline as a function of the horizon over which the utility is discounted, or in other words, we want to consider discount factors that decrease rapidly for short period delays and less rapidly for longer period delays. The discount rates which are implied by the discount factors (3.8) and which present the rates of time preference take the form

$$-\frac{\phi'(t)}{\phi(t)} = \rho + \frac{\delta}{1+\delta t}\varepsilon, \quad 0 \le t \le T. \tag{3.9}$$

We can easily notice that our rates of time preference (3.9) are smoothly declining from $\rho + \delta\varepsilon$ at $t = 0$ to $\rho$ at $t = \infty$. Hence, the desirable pattern of rates of time preference can be modelled with (3.8). Increasing $\rho$ increases our discount rates at all horizons and increasing $\delta$ raises our discount rates more at short horizons than at long horizons. Parameter $\varepsilon$ determines how close our discount rates are to a constant rate. The discount factors (3.8) are proposed for the first time by Luttmer and Mariotti (2003) and considered in utility optimization problems by Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Ekeland et. al. (2012). The discount factors (3.8) are presented in Figures 1 and 2 below.

We define the value function for the optimization problem (3.7)

$$\begin{aligned} v^{\pi,u}(t,x,\lambda) = \mathbb{E}\Big[\int_t^{\tau\wedge T} &\phi(s-t)u(s)^\gamma ds \\ &+ q\phi(T-t)(X^{\pi,u}(T))^\gamma 1_{\{\tau>T\}}|X(t)=x,\lambda(t)=\lambda,\tau>t\Big], \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K}, \end{aligned} \tag{3.10}$$

together with the optimal value function

$$v(t,x,\lambda) = v^{\pi^*,u^*}(t,x,\lambda) = \sup_{(\pi,u)\in\mathcal{A}} v^{\pi,u}(t,x,\lambda), \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K}. \tag{3.11}$$

If the time horizon $T$ is chosen so that $Pr(\tau > T)$ is negligible, then the bequest is of little importance and the investment/withdrawal problem until the retiree's death is considered.

If an exponential discount factor is applied in (3.10), then Dynamic Programming Principle can be applied to solve the optimization problem (3.11). The problem (3.11) is time consistent in the sense that if at time $t_1$ we solve (3.11) and find the optimal strategy for the period $[t_1, T]$, then this optimal strategy coincides with the optimal strategy for period $[t_2, T]$ found by solving (3.11) at time $t_2$. By classical techniques we can conclude that the optimal value function (3.11) for an exponential discount factor, let denote it by $v^{exp}$, solves the HJB equation:

$$
\begin{aligned}
& v_t^{exp}(t,x,\lambda) + \sup_{\pi,u}\left\{u^\gamma + \mathcal{L}_x^{\pi,u} v^{\exp}(t,x,\lambda)\right\} \\
& \qquad + \mathcal{L}_\lambda v^{exp}(t,x,\lambda) - \lambda v^{exp}(t,x,\lambda) = \rho v^{exp}(t,x,\lambda), \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times \mathcal{K}, \\
& v^{exp}(T,x,\lambda) = q x^\gamma, \quad (x,\lambda) \in \mathbb{R} \times \mathcal{K},
\end{aligned} \tag{3.12}
$$

and the corresponding optimal strategies $\pi^{exp}, u^{exp}$ are given by

$$
\pi^{exp}(t,x,\lambda) = -\frac{v_x^{exp}(t,x,\lambda)}{x v_{xx}^{exp}(t,x,\lambda)} \frac{\mu - r}{\sigma^2}, \quad u^{exp}(t,x,\lambda) = \left(\frac{v_x^{exp}(t,x,\lambda)}{\gamma}\right)^{\frac{1}{\gamma-1}}.
$$

The solution to (3.12) represents a special case of our general solution which we derive in the next section. We would like to point out that although the form of the HJB (3.12) is known in this case, finding a solution to (3.12) is non-trivial.

For a general discount factor (3.8), it is known that the optimization problem (3.11) is time inconsistent and Dynamic Programming Principle cannot be applied. More intuitively, if at time $t_1$, we solve (3.11) and find the optimal strategy for the period $[t_1, T]$, this optimal strategy is different from the optimal strategy for the period $[t_2, T]$ found by solving (3.11) at time $t_2$. Such an inconsistency arises since investor's preferences and discount rates change over time $[0, T]$.

For a general discount factor, the notion of an optimal strategy has to be properly defined for the time-inconsistent optimization problem (3.11). A game-theoretic approach to solving (3.11) is proposed in Björk and Murgoci (2010). Following them, we can think of the problem (3.11) as a game played by a continuum of players. Each player, indexed with variable $t$, has her own utility function and controls the wealth only over an infinitesimal period of time $[t, t + dt]$. The player $t$ has a control over the wealth at time $t$ and can freely choose a strategy at time $t$. Next, the player has to pass the wealth to the next player who has a different utility and again can freely choose a strategy. We can define the equilibrium of this game as follows.

**Definition 3.3.** *Let us consider an admissible strategy* $(\pi^*, u^*) \in \mathcal{A}$*. Choose an arbitrary point* $(t, x, \lambda) \in [0, T) \times \mathbb{R} \times \mathcal{K}$ *and any admissible strategy* $(\pi, u) \in \mathcal{A}$*. We define a new admissible strategy*

$$(\pi_h(s), u_h(s)) = \begin{cases} (\pi(s), u(s)), & t \leq s \leq (t+h) \wedge \tau, \\ (\pi^*(s), u^*(s)), & (t+h) \wedge \tau < s \leq T \wedge \tau. \end{cases}$$

*If*

$$\liminf_{h \to 0} \frac{v^{\pi^*, u^*}(t, x, \lambda) - v^{\pi_h, u_h}(t, x, \lambda)}{h} \geq 0, \quad (t, x, \lambda) \in [0, T) \times \mathbb{R} \times \mathcal{K}, \quad (3.13)$$

*then* $(\pi^*, u^*)$ *is called an equilibrium strategy and* $v^{\pi^*, u^*}(t, x, \lambda)$ *is called the equilibrium value function corresponding to the equilibrium strategy* $(\pi^*, u^*)$*.*

When the player $t$ decides on the strategy at time $t$, she should take into consideration that in the future next players will have different objectives/preferences and are likely to choose different strategies. The idea of the equilibrium strategy defined in (3.13) is that the player $t$ will not be better off by forcing the next players to choose her optimal strategy instead of letting the next players choose their best strategies. If the player at time $t$ knows that all players after her will choose a strategy $(\pi^*, u^*)$, then it is optimal for the player $t$ to choose the strategy $(\pi^*, u^*)$ at time $t$ as well. The equilibrium strategy is time consistent and the players have no incentives to deviate from the equilibrium strategy. In fact, the time consistency defined in (3.13) is a minimal requirement for rationality, see Ekeland et. al. (2012).

From Björk and Murgoci (2010) and Ekeland et. al. (2012) we expect that our equilibrium value function for the problem (3.11) should satisfy the non-local Hamilton-

Jacobi-Bellman equation

$$\begin{aligned}
& v_t(t,x,\lambda) + \sup_{\pi,u}\big\{u^\gamma + \mathcal{L}_x^{\pi,u} v(t,x,\lambda)\big\} + \mathcal{L}_\lambda v(t,x,\lambda) - \lambda v(t,x,\lambda) \\
& \quad = -\mathbb{E}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds \\
& \quad + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\} | X(t) = x, \lambda(t) = \lambda, \tau > t\Big], \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times \mathcal{K}, \\
& v(T,x,\lambda) = qx^\gamma, \quad (x,\lambda) \in \mathbb{R} \times \mathcal{K},
\end{aligned} \tag{3.14}$$

where the equilibrium strategy $(\pi^*, u^*)$ realizes the supremum in left hand side of equation (3.14), i.e.

$$(\pi^*, u^*) = arg\ max\ _{\pi,u} \big\{u^\gamma + \mathcal{L}_x^{\pi,u} v(t,x,\lambda)\big\}. \tag{3.15}$$

The operators $\mathcal{L}_x^{\pi,u}$, $\mathcal{L}_\lambda$ are given by (2.6), (3.4).[3]

We can prove the following verification theorem.

**Theorem 3.1.** *Let (A1)-(A5) hold. Assume there exists a function* $v \in \mathcal{C}([0,T] \times \mathbb{R} \times \mathcal{K}) \cap \mathcal{C}^{1,2,2}([0,T) \times \mathbb{R} \times \mathcal{K})$ *and an admissible strategy* $(\pi^*, u^*) \in \mathcal{A}$ *which solve the HJB equation* (3.14)*. In addition, assume that the sequence*

$$\Big\{v(\mathcal{T}, X^{\pi^*,u^*}(\mathcal{T}), \lambda(\mathcal{T})),\ \mathcal{T}\ is\ an\ \mathbb{F}-stopping\ time, \mathcal{T} \in [0,T]\Big\}$$

*is uniformly integrable. The strategy* $(\pi^*, u^*) \in \mathcal{A}$ *is an equilibrium strategy and* $v(t,x,\lambda) = v^{\pi^*,u^*}(t,x,\lambda)$ *is the equilibrium value function corresponding to* $(\pi^*, u^*)$.

Details can be found in the Appendix.

[3] If we choose the dynamics of the mortality intensity with jumps (2.5), we obtain the same HJB (3.14) but with a different operator $\mathcal{L}_\lambda$, see (2.7).

# 4 The solution to the optimization problem

Solving the optimization problem (3.11) and the HJB equation (3.14) is challenging due to the non-local term in (3.14), see Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Ekeland et. al. (2012) and Dong and Sircar (2014). In order to solve the non-local HJB equation (3.14), we use the expansion method suggested in Dong and Sircar (2014). The expansion method allows us to find the first order approximations of the equilibrium value function and the equilibrium strategies. This method provides a good approximation when the part of the hyperbolic discounting in the general discount factor (3.8) is negligibly small.

We assume that the discount factors (3.8) are derived from exponential discount factors by adding a small amount of hyperbolic discounting, i.e. we consider the non-exponential discount factors of the form

$$\phi(t) = e^{-\rho t}\frac{1}{(1+\delta t)^{\varepsilon}} = e^{-\rho t-\varepsilon\ln(1+\delta t)}, \quad \varepsilon\to 0, \quad 0\le t\le T. \tag{4.1}$$

We can expand the discount factors to the first order at $\varepsilon = 0$ and we get the approximation

$$\phi(t) = e^{-\rho t}(1+\vartheta(t)\varepsilon) + o(\varepsilon^2), \quad \varepsilon\to 0, \quad 0\le t\le T, \tag{4.2}$$

where

$$\vartheta(t) = -\ln(1+\delta t), \quad 0\le t\le T.$$

Since $\varepsilon\to 0$, the discount factors (4.2) are positive.

In Figures 1 and 2 we can see three discounting functions: exponential, non-exponential (4.1) and approximation (4.2). As already discussed, by using the discounting function (4.1) we can model discount factors that decrease rapidly for short period delays and less rapidly for longer period delays which is clearly observed in Fig-

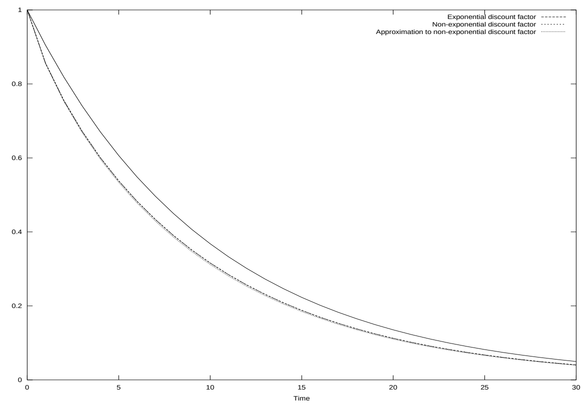


Figure 1: The discounting functions for $\rho = 0.1, \delta = 2, \varepsilon = 0.05$: exponential (solid line), non-exponential (4.1) and approximation (4.2) (dashed lines).

ures 1 and 2. Moreover, Figures 1 and 2 show that the first order approximation (4.2) to the true non-exponential discounting function (4.1) is very good for $\varepsilon = 0.05, 0.1$.

Let us now study the HJB equation (3.14). In the sequel, the conditional expectation $\mathbb{E}[\cdot|X(t) = x, \lambda(t) = \lambda, \tau > t]$ is denoted as $\mathbb{E}_{t,x,\lambda}[\cdot]$. The first order conditions w.r.t $u$ and $\pi$ yield the following candidates for the optimal strategies

$$\pi^*(t) = -\frac{v_x(t,x,\lambda)}{xv_{xx}(t,x,\lambda)}\frac{\mu - r}{\sigma^2}, \quad u^*(t) = \left(\frac{v_x(t,x,\lambda)}{\gamma}\right)^{\frac{1}{\gamma-1}}. \tag{4.3}$$

Substituting the strategies (4.3) into the HJB equation (3.14), we obtain the equation

$$\begin{aligned}
& v_t(t,x,\lambda) + \big(xr + a(\alpha\hat{x}) - c^*\big)v_x(t,x,\lambda) - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{v_x^2(t,x,\lambda)}{v_{xx}(t,x,\lambda)} \\
& \quad + \Big(\frac{v_x(t,x,\lambda)}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}}(1-\gamma) + \mathcal{L}_\lambda v(t,x,\lambda) - \lambda v(t,x,\lambda) \\
& = -\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds \\
& \quad + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\}\Big], \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times \mathcal{K}, \\
& v(T,x,\lambda) = qx^\gamma, \quad (x,\lambda) \in \mathbb{R} \times \mathcal{K},
\end{aligned} \tag{4.4}$$

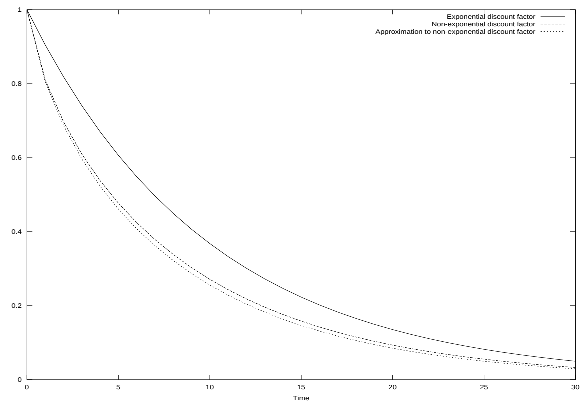


Figure 2: The discounting functions for $\rho = 0.1, \delta = 2, \varepsilon = 0.1$: exponential (solid line), non-exponential (4.1) and approximation (4.2) (dashed lines).

where $\pi^*$ and $u^*$ are given by (4.3). Since we introduce the first order expansion for the discounting function (4.1), it is reasonable to introduce the first order expansions for the value function and the strategies

$$\begin{aligned} v(t,x,\lambda) &= v^0(t,x,\lambda) + v^1(t,x,\lambda)\varepsilon + o(\varepsilon^2), \quad \varepsilon \to 0, \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times \mathcal{K}, \\ \pi^*(t) &= \pi^{0,*}(t) + \pi^{1,*}(t)\varepsilon + o(\varepsilon^2), \quad \varepsilon \to 0, \quad 0 \le t \le T, \\ u^*(t) &= u^{0,*}(t) + u^{1,*}(t)\varepsilon + o(\varepsilon^2), \quad \varepsilon \to 0, \quad 0 \le t \le T. \end{aligned} \tag{4.5}$$

Our goal is to find $v^0, v^1$ together with $\pi^{0,*}, \pi^{1,*}$ and $u^{0,*}, u^{1,*}$.

If (4.5) holds, then we also have

$$\begin{aligned} &\Big(v_x^0(t,x,\lambda) + v_x^1(t,x,\lambda)\varepsilon + o(\varepsilon^2)\Big)^{\frac{\gamma}{\gamma-1}} \\ &= \big(v_x^0(t,x,\lambda)\big)^{\frac{\gamma}{\gamma-1}} + \frac{\gamma}{\gamma-1}\big(v_x^0(t,x,\lambda)\big)^{\frac{1}{\gamma-1}} v_x^1(t,x,\lambda)\varepsilon + o(\varepsilon^2), \\ &\big(v_x^0(t,x,\lambda) + v_x^1(t,x,\lambda)\varepsilon + o(\varepsilon^2)\big)^2 \big(v_{xx}^0(t,x,\lambda) + v_{xx}^1(t,x,\lambda)\varepsilon + o(\varepsilon^2)\big)^{-1} \\ &= \frac{\big(v_x^0(t,x,\lambda)\big)^2}{v_{xx}^0(t,x,\lambda)} \\ &\quad + \Big(2\frac{v_x^0(t,x,\lambda)}{v_{xx}^0(t,x,\lambda)} v_x^1(t,x,\lambda) - \Big(\frac{v_x^0(t,x,\lambda)}{v_{xx}^0(t,x,\lambda)}\Big)^2 v_{xx}^1(t,x,\lambda)\Big)\varepsilon + o(\varepsilon^2), \end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[\int_t^{\tau\wedge T}\phi'(s-t)(u^*(s))^\gamma ds + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau>T\}\Big] \\
&= \quad -\rho v^0(t,x,\lambda) - \rho v^1(t,x,\lambda)\varepsilon \\
&\qquad +\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds \\
&\qquad +q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma \mathbf{1}\{\tau>T\}\Big]\varepsilon + o(\varepsilon^2). \qquad (4.7)
\end{aligned}
$$

Formula (4.7) is far from obvious and is derived in the Appendix in (7.24). After substituting our expansions (4.5)-(4.7) into the HJB equation (4.4), we collect the terms which are independent of $\varepsilon$, those which are proportional to $\varepsilon$ and those of order $o(\varepsilon^2)$. Subsequently, we obtain two equation systems which characterize the functions $v^0$ and $v^1$:

$$
\begin{aligned}
&v^0_t(t,x,\lambda) + \big(xr + a(\alpha\hat{x}) - c^*\big)v^0_x(t,x,\lambda) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{(v^0_x(t,x,\lambda))^2}{v^0_{xx}(t,x,\lambda)} \\
&\quad +\Big(\frac{v^0_x(t,x,\lambda)}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}}(1-\gamma) + \mathcal{L}_\lambda v^0(t,x,\lambda) - \lambda v^0(t,x,\lambda) \\
&= \rho v^0(t,x,\lambda), \quad (t,x,\lambda)\in[0,T)\times\mathbb{R}\times\mathcal{K}, \\
&v^0(T,x,\lambda) = qx^\gamma, \quad (x,\lambda)\in\mathbb{R}\times\mathcal{K}, \qquad (4.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&v^1_t(t,x,\lambda) + \Big(xr + a(\alpha\hat{x}) - c^* - \frac{(\mu-r)^2}{\sigma^2}\frac{v^0_x(t,x,\lambda)}{v^0_{xx}(t,x,\lambda)} - \Big(\frac{v^0_x(t,x,\lambda)}{\gamma}\Big)^{\frac{1}{\gamma-1}}\Big)v^1_x(t,x,\lambda) \\
&\quad +\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\Big(\frac{v^0_x(t,x,\lambda)}{v^0_{xx}(t,x,\lambda)}\Big)^2 v^1_{xx}(t,x,\lambda) + \mathcal{L}_\lambda v^1(t,x,\lambda) - \lambda v^1(t,x,\lambda) \\
&= \rho v^1(t,x,\lambda) - \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds \\
&\quad +q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma\mathbf{1}\{\tau>T\}\Big], \quad (t,x,\lambda)\in[0,T)\times\mathbb{R}\times\mathcal{K}, \\
&v^1(T,x,\lambda) = 0, \quad (x,\lambda)\in\mathbb{R}\times\mathcal{K}. \qquad (4.9)
\end{aligned}
$$

If we compare our equation (4.8) with the HJB equation (3.12) for $v^{exp}$ after substituting the optimal strategies $\pi^{exp}$, $u^{exp}$, we can notice that the function $v^0$ is the optimal value function for the optimization problem (3.11) with exponential discounting. This sounds reasonable, because for $\varepsilon = 0$ our discounting function reduces to exponential function. In the Appendix, see (7.22)-(7.23), we show that $v^0$ is the value function (3.10) with exponential discounting under the strategies $(\pi^{0,*}, u^{0,*})$. Hence, our zeroth order strategies $(\pi^{0,*}, u^{0,*})$ are the optimal strategies for the optimization problem (3.11) with exponential discounting. By (3.12), we get the zeroth order strategies

$$\pi^{0,*}(t) = -\frac{v^0_x(t,x,\lambda)}{xv^0_{xx}(t,x,\lambda)}\frac{\mu - r}{\sigma^2}, \quad u^{0,*}(t) = \left(\frac{v^0_x(t,x,\lambda)}{\gamma}\right)^{\frac{1}{\gamma-1}}. \tag{4.10}$$

Let us assume

$$\begin{aligned} v^0(t,x,\lambda) &= (x+g(t))^\gamma f(t,\lambda), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times \mathcal{K}, \\ v^1(t,x,\lambda) &= (x+G(t))^\gamma F(t,\lambda), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times \mathcal{K}. \end{aligned}$$

The terms $g(t)$ and $G(t)$ arise due to the income generated by the life time annuity.

Derivation of the functions $f$ and $g$ is rather standard. Plugging the above expression of $v^0(t,x,\lambda)$ in (4.8), we obtain the equations:

$$\begin{aligned} & f_t(t,\lambda) + \mathcal{L}_\lambda f(t,\lambda) - \left(\lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right) f(t,\lambda) \\ & \quad +(1-\gamma)(f(t,\lambda))^{\gamma/(\gamma-1)} = 0, \quad (t,\lambda) \in [0,T) \times \mathcal{K}, \\ & f(T,\lambda) = q, \quad \lambda \in \mathcal{K}, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} & g_t(t) - g(t)r + a(\alpha\hat{x}) - c^* = 0, \quad t \in [0,T), \\ & g(T) = 0. \end{aligned} \tag{4.12}$$

Deriving the functions $F$ and $G$ is much more complex, because we have to calculate the expectation in (4.9). Compared to equation (4.4), the non-local term in the equation for $v^1$ now involves strategies determined by $v^0$. i.e. the zeroth order strategies (4.10). This is the key point that simplifies the calculations and the derivation of the solution to our optimization problem. In the Appendix, see (7.17) and (7.21), we show that

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds \\
&\qquad +q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma\mathbf{1}\{\tau>T\}\Big] \\
&= \quad (x+g(t))^\gamma Q(t,\lambda), \quad (t,\lambda)\in[0,T]\times\mathcal{K}, \qquad (4.13)
\end{aligned}$$

where

$$\begin{aligned}
&Q(t,\lambda) \\
&= \int_t^T\vartheta'(s-t)P^s(t,\lambda)ds+q^{\frac{1}{1-\gamma}}\vartheta'(T-t)P^T(t,\lambda), \quad (t,\lambda)\in[0,T]\times\mathcal{K}, \quad (4.14)
\end{aligned}$$

and the function $P^s$ is obtained as a unique solution to the PDE

$$\begin{aligned}
&P^s_t(t,\lambda)+\mathcal{L}_\lambda P^s(t,\lambda)-\big(\lambda+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2} \\
&\qquad +\gamma f(t,\lambda)^{\frac{1}{\gamma-1}}\big)P^s(t,\lambda)=0, \quad t\in[0,s) \\
&P^s(s,\lambda)=f(s,\lambda)^{\frac{\gamma}{\gamma-1}}, \quad \lambda\in\mathcal{K}. \qquad (4.15)
\end{aligned}$$

By standard arguments, we can now derive equations for the functions $F$ and $G$. We end up with the equations:

$$\begin{aligned}
&F_t(t,\lambda)+\mathcal{L}_\lambda F(t,\lambda)-\big(\lambda+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2} \\
&\qquad +\gamma f(t,\lambda)^{\frac{1}{\gamma-1}}\big)F(t,\lambda)+Q(t,\lambda)=0, \quad (t,\lambda)\in[0,T)\times\mathcal{K}, \\
&F(T,\lambda)=0, \quad \lambda\in\mathcal{K}, \qquad (4.16)
\end{aligned}$$

and

$$\begin{aligned}
&G_t(t) - G(t)r + a(\alpha\hat{x}) - c^* = 0, \quad t \in [0,T), \\
&G(T) = 0. \qquad (4.17)
\end{aligned}$$

It is clear that there exists a unique solution to equations (4.12), (4.17) which is of the form

$$\begin{aligned}
g(t) &= G(t) = \int_t^T (a(\alpha\hat{x}) - c^*)e^{-r(s-t)}ds \\
&= (a(\alpha\hat{x}) - c^*)\frac{1-e^{-r(T-t)}}{r}, \quad 0 \le t \le T. \qquad (4.18)
\end{aligned}$$

The existence of solutions to equations (4.11), (4.16) is not trivial. We can prove the following result.

**Proposition 4.1.** *Let (A2)-(A3), (A5) hold. There exists unique solutions* $f, F \in \mathcal{C}([0,T]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,T)\times\mathcal{K})$ *to equations* (4.11), (4.16). *The functions $f$ and $F$ are bounded. Moreover, the function $f$ is uniformly bounded away from zero, i.e.* $f(t,\lambda) \ge K > 0$, $(t,\lambda) \in [0,T]\times\mathcal{K}$, *and the function $F$ is non-positive, i.e.* $F(t,\lambda) \le 0, (t,\lambda) \in [0,T]\times\mathcal{K}$.

Recalling the formulas for the candidate optimal strategies (4.3) and the expansions (4.5), (4.6), we can derive the first order approximations to the equilibrium strategies and the equilibrium value function.

**Theorem 4.1.** *Let (A1)-(A5) hold and $\varepsilon \to 0$ in the discount factors* (4.2). *Consider the HJB equation* (3.14). *The first order approximation to the function $v$ which solves the HJB equation takes the form*

$$v^{0,*}(t,x,\lambda) + v^{1,*}(t,x,\lambda)\varepsilon = (x+g(t))^\gamma\big(f(t,\lambda) + F(t,\lambda)\varepsilon\big), \quad (t,x,\lambda) \in [0,T]\times\mathbb{R}\times\mathcal{K},$$

*and the first order approximations to the equilibrium investment and consumption strate-*

*gies are of the form*

$$\begin{aligned}
\tilde{\pi}^*(t,x,\lambda) &= \frac{\mu-r}{\sigma^2(1-\gamma)}\frac{x+g(t)}{x}, \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K},\\
\tilde{u}^*(t,x,\lambda) &= (x+g(t))f(t,\lambda)^{\frac{1}{\gamma-1}}\\
&\quad\cdot\Big(1-\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon\Big), \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K}, \qquad (4.19)
\end{aligned}$$

*where the functions* $g,\ f,\ F$ *solve equations* (4.18)*,* (4.11)*,* (4.16)*.*

It remains to prove that the strategies (4.19) are admissible. The strategies are Markov and progressively measurable provided that the solution $X^{\tilde{\pi}^*,\tilde{u}^*}$ exists. Let us investigate the wealth process under the first order equilibrium strategies. If we substitute the strategies from Theorem 4.1 into the SDE (3.3) which describes the retiree's wealth process, we get the dynamics

$$\begin{aligned}
dX^{\tilde{\pi}^*,\tilde{u}^*}(t) &= \big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)\frac{\mu-r}{\sigma^2(1-\gamma)}\big(\mu dt+\sigma dW_m(t)\big)\\
&+\Big(\big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)-g(t)-(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t))\frac{\mu-r}{\sigma^2(1-\gamma)}\Big)rdt+a(\alpha\hat{x})dt-c^*dt\\
&-\big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)f(t,\lambda(t))^{\frac{1}{\gamma-1}}\Big(1-\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon\Big)dt, \quad 0\le t\le T. \qquad (4.20)
\end{aligned}$$

Let

$$\tilde{\mu}=r+\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}, \quad \tilde{\sigma}=\frac{\mu-r}{\sigma(1-\gamma)}, \quad \tilde{f}(t,\lambda)=f(t,\lambda)^{\frac{1}{\gamma-1}}\Big(1-\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon\Big).$$

If we recall the equation for $g$, see (4.12), then the dynamics (4.20) can be rewritten in the form

$$\begin{aligned}
d\big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big) &= \big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)dZ(t), \quad 0\le t\le T,\\
dZ(t) &= \big(\tilde{\mu}-\tilde{f}(t,\lambda(t))\big)dt+\tilde{\sigma}dW_m(t), \quad 0\le t\le T. \qquad (4.21)
\end{aligned}$$

We can conclude that the process $X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)$ is a stochastic exponential of $Z$ and owns the solution

$$X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)=((1-\alpha)\hat{x}+g(0))e^{(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)t-\int_0^t\tilde{f}(s,\lambda(s))ds+\tilde{\sigma}W_m(t)},\quad 0\leq t\leq T. \quad (4.22)$$

Hence, the SDE (3.3) under $(\tilde{\pi}^*,\tilde{u}^*)$ has a unique solution. The retiree's wealth process under the first order equilibrium strategies (4.19) takes the form

$$X^{\tilde{\pi}^*,\tilde{u}^*}(t)=((1-\alpha)\hat{x}+g(0))e^{(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)t-\int_0^t\tilde{f}(s,\lambda(s))ds+\tilde{\sigma}W_m(t)}-g(t),\quad 0\leq t\leq T.$$

Since the function $\tilde{f}$ is bounded, we conclude that the process $X^{\tilde{\pi}^*,\tilde{u}^*}$ is square integrable and

$$\mathbb{E}\Big[\int_0^T|\tilde{\pi}^*(t)X^{\tilde{\pi}^*,\tilde{u}^*}(t)|^2dt+\int_0^T|\tilde{u}^*(t)|^2dt\Big]<\infty.$$

Consequently, the strategy $(\tilde{\pi}^*,\tilde{u}^*)$ is square integrable. Finally, we show that $\tilde{u}^*(t)\geq 0,\ 0\leq t\leq T,\ X^{\tilde{\pi}^*,\tilde{u}^*}(T)\geq 0$. We can notice that

$$(1-\alpha)\hat{x}+g(0)=\hat{x}-\int_0^T c^*e^{-rt}dt+\alpha\hat{x}\Big(\frac{\int_0^T e^{-rt}dt}{a}-1\Big), \quad (4.23)$$

where $a$ denotes the present value of an annuity paying 1 for lifetime but not longer than $T$ years, i.e. $a=\alpha\hat{x}/a(\alpha\hat{x})$. Clearly, we have $a<\int_0^T e^{-rt}dt$. Since (A4) holds, we get $(1-\alpha)\hat{x}+g(0)\geq 0$. Consequently, $X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\geq 0,\ 0\leq t\leq T$, by (4.22) and $\tilde{u}^*(t)\geq 0,\ 0\leq t\leq T,\ X^{\tilde{\pi}^*,\tilde{u}^*}(T)\geq 0$, by (4.19) and Proposition 4.1.

We conclude with the following result.

**Theorem 4.2.** *Consider the optimization problem* (3.10)-(3.11) *with the discount factors* (4.2)*. Let the assumptions of Theorem 4.1 hold. The strategy* $(\tilde{\pi}^*,\tilde{u}^*)$ *is admissible, i.e.* $(\tilde{\pi}^*,\tilde{u}^*)\in\mathcal{A}$*, and the sequence*

$$\Big\{v^0(\mathcal{T},X^{\tilde{\pi}^*,\tilde{u}^*}(\mathcal{T}),\lambda(\mathcal{T}))+v^1(\mathcal{T},X^{\tilde{\pi}^*,\tilde{u}^*}(\mathcal{T}),\lambda(\mathcal{T}))\varepsilon,$$
$$\mathcal{T}\ \textit{is an}\ \mathbb{F}-\textit{stopping time},\mathcal{T}\in[0,T]\Big\},$$

*is uniformly integrable. Consequently, in the view of the results of Proposition 4.1, Theorems 3.1, 4.1,* $v^{0,*}(t,x,\lambda) + v^{1,*}(t,x,\lambda)\varepsilon$ *is the first order approximation to the equilibrium value function and* $(\tilde{\pi}^*, \tilde{u}^*)$ *gives the first order approximations to the equilibrium investment and consumption strategies for the optimization problem* (3.10)-(3.11).

Admissability of $(\tilde{\pi}^*, \tilde{u}^*)$ has already been shown. Uniform integrability is discussed in the Appendix.[4]

One may ask whether there is an optimal choice of $\alpha$ at the retirement date $t = 0$? The first order approximation to the equilibrium value function gives us

$$v(0,(1-\alpha)\hat{x},\lambda) = \big((1-\alpha)\hat{x} + g(0)\big)^{\gamma}\big(f(0,\lambda) + F(0,\lambda)\varepsilon\big) + o(\varepsilon^2), \quad \varepsilon \to 0, \quad (4.24)$$

Since $(1-\alpha)\hat{x} + g(0)$ is given by (4.23), we can conclude that $\alpha = 1$ is the optimal choice. Let us remark that although the function $F$ is non-positive, the expansion (4.24) holds for sufficiently small $\varepsilon$ and $f(0,\lambda) + F(0,\lambda)\varepsilon > 0$ as the function $f$ is positive. Notice that even for $\alpha = 1$ we are interested in finding the optimal investment and consumption strategies.

# 5 Interpretation of the equilibrium strategy and numerical example

In this section we comment on the derived first order equilibrium consumption and investment strategies (4.19). Some properties of the equilibrium consumption and investment strategies can de directly derived from the formulas (4.19), other properties have to be established by considering a numerical example. For the purpose of investigating the equilibrium strategy $(\tilde{\pi}^*, \tilde{u}^*)$, we assume that the dynamics of the mortality

[4]If we choose the dynamics of the mortality intensity with jumps (2.5), we will derive the same results as above. Only a different generator $\mathcal{L}_\lambda$ for the mortality intensity process $\lambda$ would arise in equations (4.11), (4.16). Proving existence and uniqueness of solutions to equations (4.11), (4.16) would be a bit more difficult if the dynamics (2.5) were used, see Delong and Klüppelberg (2008).

intensity is modelled with a geometric Brownian motion, i.e. the mortality intensity process satisfies the SDE

$$d\lambda(t) = \mu_\lambda \lambda(t)dt + \sigma_\lambda \lambda(t)dW_\lambda(t).$$

Clearly, the mortality intensity is not bounded, as we require when solving our optimization problem, but we can set a sufficiently high cap on the process and assume that the mortality intensity process is bounded.

In our numerical example we consider the set of parameters presented in Table 1.

| The parameter | The value | The parameter | The value |
|---:|---:|---:|---:|
| $\mu$ | 0.1 | $\sigma$ | 0.2 |
| $r$ | 0.05 | $\gamma$ | 0.1 |
| $\mu_\lambda$ | 0.1 | $\sigma_\lambda$ | 0.05 |
| $\rho$ | 0.1 | $\delta$ | 2 |
| $T$ | 30 | $q$ | 1 |

Table 1: Values of the parameters considered in the numerical example.

The probability that the retiree survives 30 years is estimated at the level of $1.7 * 10^{-6}$. If we choose a deterministic mortality intensity, i.e. we set $\sigma_\lambda = 0$, then the probability that the retiree survives 30 years is equal to $3.5 * 10^{-8}$. Hence, if a random noise is added into the mortality intensity as in (5.1), longevity risk arises and the retiree is expected to live longer.

First, let us investigate the dependence of the equilibrium consumption and investment strategy on the level of the bought annuity. The level of the annuity bought at time $t = 0$, denoted by $a(\alpha\hat{x})$, only arises in the function $g$. We assume that the minimal consumption level $c^*$ is specified. If $a(\alpha\hat{x}) = c^*$, then $g(t) = 0$. If $a(\alpha\hat{x}) > c^*$, then $g(t) > 0$ and the retiree consumes at a higher rate and invests more aggressively. In this case, a negative wealth is possible at some intermediate points of time and the retiree is not afraid that she will not be able to sustain the minimum consumption level $c^*$. If $a(\alpha\hat{x}) < c^*$, then $g(t) < 0$ and the retiree consumes at a lower rate and invests more

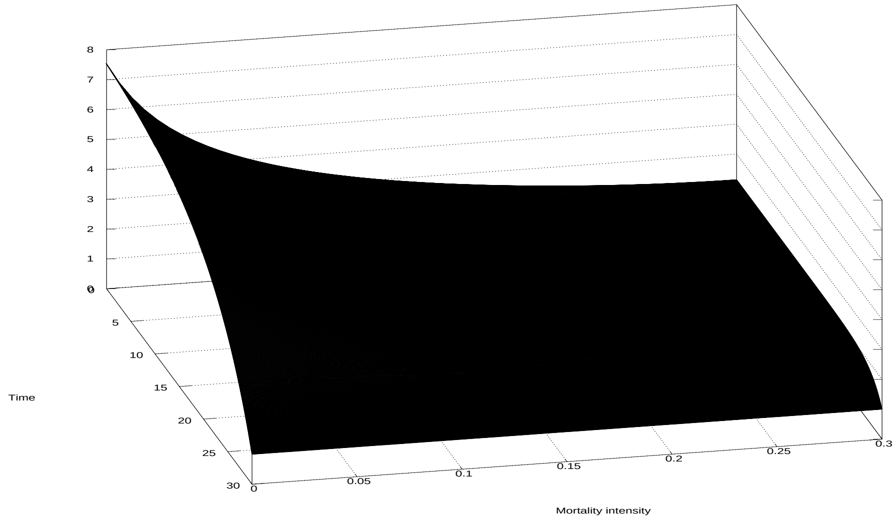


Figure 3: The function $f$ which solves the PDE (4.11).

conservatively. In this case, the retiree's wealth is always positive and the retiree must act in the financial market so that she will be able to consume at least the minimum consumption $c^*$.

The equilibrium investment strategy does not depend on the mortality intensity. This is reasonable since the retiree simply aims at maximizing her total utility by implementing the best asset allocation. In contrast, the consumption strategy must depend on the mortality intensity. The dependence of the equilibrium consumption strategy on the mortality intensity is investigated with a numerical example. We assume that $x + g(t) = 1, 0 \leq t \leq T$, so that the effect of mortality is only studied.

First, we solve the PDEs (4.11), (4.16). The PDEs are solved by using explicit and implicit finite difference methods. The functions $f$ and $F$ are presented in Figures 3 and 4.

The function $f$ defines the zeroth order approximation to the equilibrium value function, hence it is the dominating term in the equilibrium value function. We can observe that the higher the mortality intensity, the lower the value of the function $f$

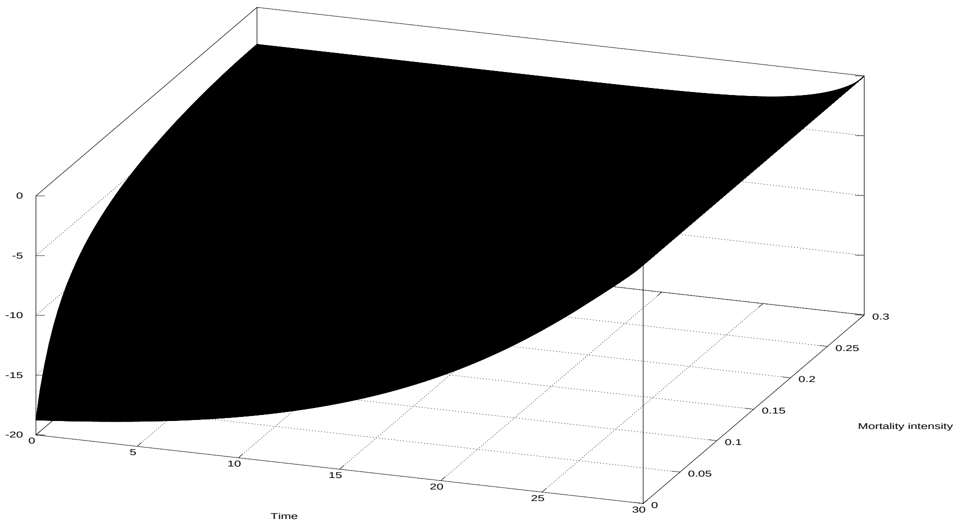


Figure 4: The function $F$ which solves the PDE (4.16).

and, consequently, the lower the equilibrium value function. This result agrees with our intuition since the retiree with a higher mortality intensity has a shorter life expectancy and the discounted value of the future consumption is lower.

The first order equilibrium consumption strategy is presented in Figure 5. In accordance with our intuition, the higher the mortality intensity, the higher the consumption rate. Obviously, the retiree with a higher mortality intensity should consume more in order to exhaust the available funds before the death which is expected to come sooner. The impact of the mortality intensity on the consumption is stronger in the first years since at that times the bequest motive does not play any role (due to negligible probability of surviving the next 30 years).

One of the goal of this paper is to consider longevity risk and investigate its impact on consumption and investment. We have already noticed that positive parameter $\sigma_\lambda$ in the dynamics of the mortality intensity (5.1), i.e. stochastic mortality intensity process, introduces longevity risk in our model. In Figures 6 and 7 we compare the equilibrium consumption strategies for $\sigma_\lambda = 0.05$ and $\sigma_\lambda = 0.3$. If we consider the stochastic

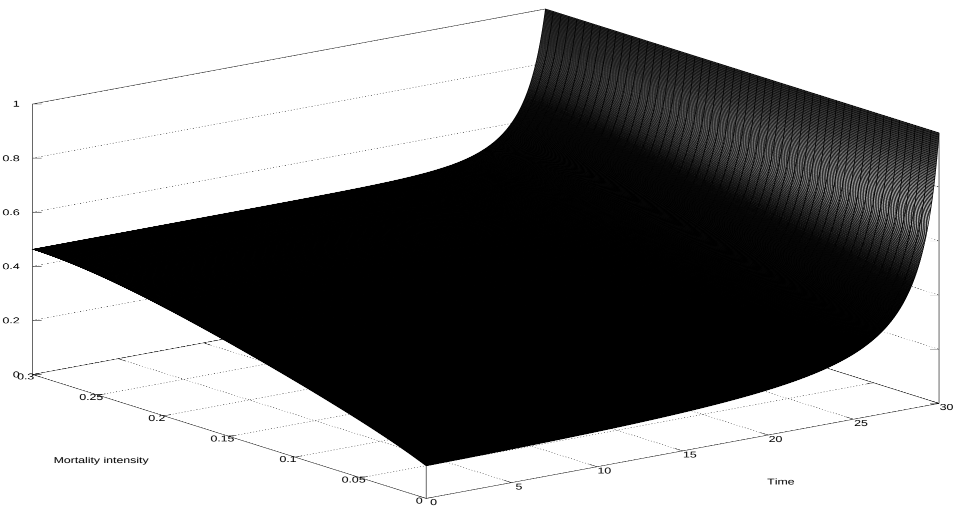


Figure 5: The first order equilibrium consumption strategy (4.19) for $\varepsilon = 0.05$.

mortality intensity process with $\sigma_\lambda = 0.3$, instead of the base value $\sigma_\lambda = 0.05$, then the the probability that the retiree survives 30 years increases from $1.7 * 10^{-6}$ to 0.037. Consequently, under a higher volatility of the stochastic mortality intensity process (5.1), the retiree has a longer life expectancy and she should consume less in order not to exhaust the available funds too soon. We can conclude that the higher the longevity risk (the higher the volatility $\sigma_\lambda$ of the stochastic mortality intensity process (5.1)), the lower the consumption rate. We can notice that the decrease in the consumption caused by a longer life expectancy is more severe for higher mortality intensity levels. This sounds reasonable since for higher mortality intensity levels the consumption rate is higher, as we have already observed, and the consumption rate should be more adjusted if the life expectancy increases.

Finally, we investigate the impact of perturbing exponential discounting with a small amount of hyperbolic discounting on investment and consumption strategies. The optimal investment strategy is the same for the time-inconsistent optimization problem ($\varepsilon > 0$) as for the time-consistent optimization problem ($\varepsilon = 0$). The optimal con-

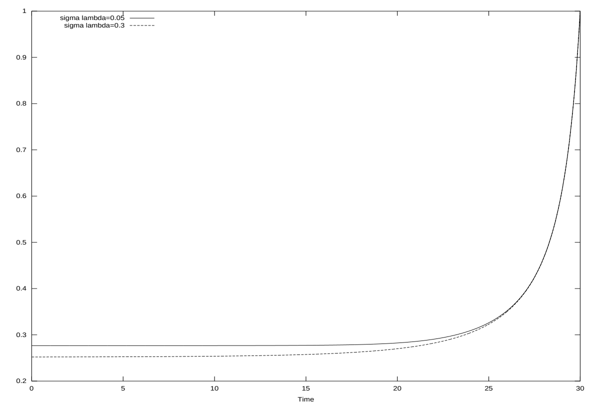


Figure 6: The first order equilibrium consumption strategy (4.19) for $\varepsilon = 0.05$ and $\lambda = 0.09$: $\sigma_\lambda = 0.05$ (solid line) and $\sigma_\lambda = 0.3$ (dashed line).

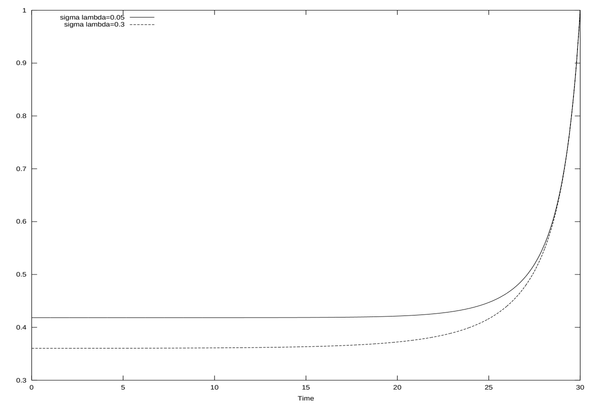


Figure 7: The first order equilibrium consumption strategy (4.19) for $\varepsilon = 0.05$ and $\lambda = 0.21$: $\sigma_\lambda = 0.05$ (solid line) and $\sigma_\lambda = 0.3$ (dashed line).

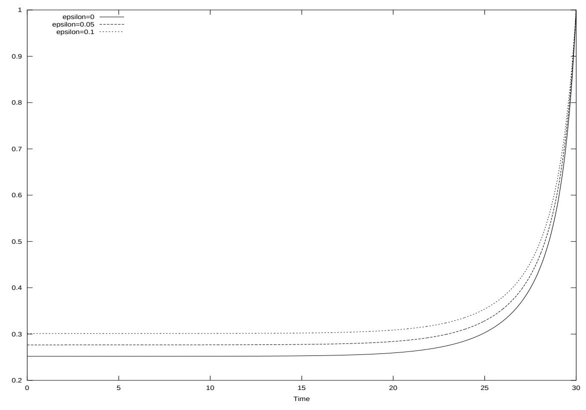


Figure 8: The first order equilibrium consumption strategy (4.19) for $\lambda = 0.09$: $\varepsilon = 0$ (solid line), $\varepsilon = 0.05$, $\varepsilon = 0.1$ (dashed lines).

sumption strategy is however adjusted in the case of the time-inconsistent optimization problem ($\varepsilon > 0$) compared with the time-consistent optimization problem ($\varepsilon = 0$). The adjustment takes the form

$$\frac{\tilde{u}^*(t) - u^{0,*}(t)}{u^{0,*}(t)} = -\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon \tag{5.1}$$

From Proposition 4.1 we know that the function $f$ is positive and the function $F$ is negative. Consequently, the first order adjustment (5.1) of the optimal time-consistent consumption strategy for the time-inconsistent problem is positive. The sign of the first order adjustment agrees with our intuition. If exponential discounting is perturbed by adding hyperbolic discounting, then the discount rates applied to the future consumption streams are higher, see Figures 1 and 2, and the retiree prefers to consume at a higher rate since she is not willing to postpone her consumption. The higher parameter $\varepsilon$ in the discounting function (4.1), i.e. the higher the amount of hyperbolic discounting added to exponential discounting, the higher the consumption rate. This is observed in Figure 7. We still assume that $x + g(t) = 1$.

The first order adjustment for the equilibrium consumption strategy (5.1) depends on the mortality intensity. We expect that the higher mortality intensity, the lower

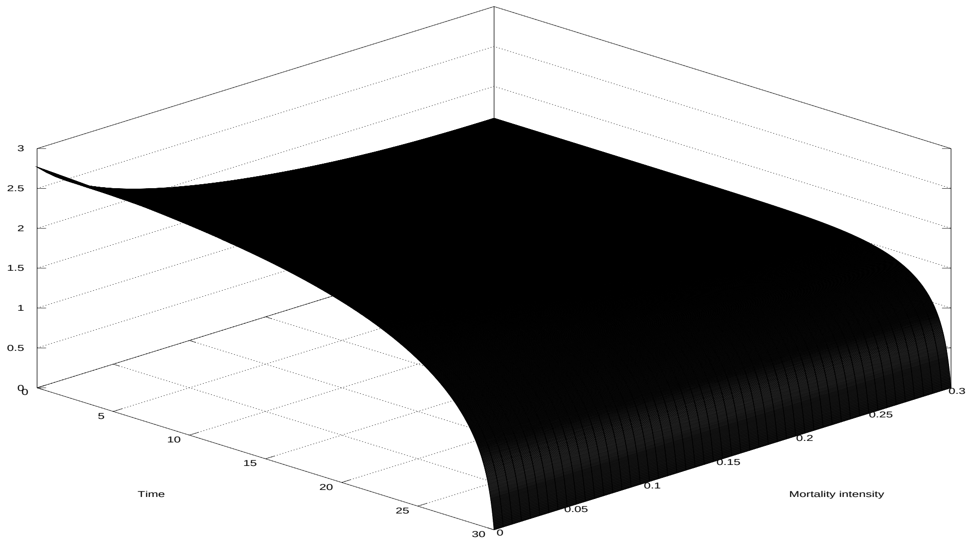


Figure 9: The first order adjustment for the equilibrium consumption strategy (5.1) for $\varepsilon = 1$.

the first order adjustment since for high mortality intensity levels the consumption is already high, as we have already observed, and the retiree is expected to die soon so the effect of increasing the discount rates is smaller. Figure 8 confirms our intuition. For the same reason the function $F$, which defines the first order adjustment for the equilibrium value function, is an increasing function of the mortality intensity level, see Figure 4.

# 6 Conclusion

In this paper we have studied a version of the Merton problem for a retiree in which we combine four important aspects: asset allocation, sustainable withdrawal, longevity risk and non-exponential discounting. We have derived a non-local HJB equation which characterizes the equilibrium investment and consumption strategy for our time-inconsistent optimization problem and we have solved the non-local HJB equation by applying expansion techniques. We have found explicit formulas for the first order approximations to the equilibrium consumption and investment strategies.

# 7 Appendix

Unless otherwise stated, we consider $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathcal{K}$.

**Proof of Theorem 3.1:** We follow the ideas from Björk and Murgoci (2010) and Ekeland et. al. (2012).

1. First we prove that a function $v^*$ which solves the HJB equation (3.14) is the value function (3.10) corresponding to the strategy $(\pi^*, u^*)$, i.e. we have the relation

$$\begin{aligned} v^*(t,x,\lambda) &= v^{\pi^*,u^*}(t,x,\lambda) \\ &= \mathbb{E}_{t,x,\lambda}\left[\int_t^{\tau\wedge T} \phi(s-t)(u^*(s))^\gamma ds + q\phi(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}_{\{\tau>T\}}\right], \end{aligned} \tag{7.1}$$

where $(\pi^*, u^*)$ is given by (3.14)-(3.15). We introduce a localizing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \to T,\ n \to \infty$. By Itô's formula we get

$$\begin{aligned} &\mathbb{E}_{t,x,\lambda}\Big[v^*(T\wedge\tau_n, X^{\pi^*,u^*}(T\wedge\tau_n), \lambda(T\wedge\tau_n))\mathbf{1}\{\tau > T\wedge\tau_n\}\Big] = v^*(t,x,\lambda) \\ &\quad +\mathbb{E}_{t,x,\lambda}\Big[\int_t^{T\wedge\tau_n\wedge\tau} v_t^*(s, X^{\pi^*,u^*}(s), \lambda(s))ds + \int_t^{T\wedge\tau_n\wedge\tau} \mathcal{L}_x^{\pi^*,u^*} v^*(s, X^{\pi^*,u^*}(s), \lambda(s))ds \\ &\quad + \int_t^{T\wedge\tau_n\wedge\tau} \mathcal{L}_\lambda v^*(s, X^{\pi^*,u^*}(s), \lambda(s))ds - \int_t^{T\wedge\tau_n\wedge\tau} v^*(s, X^{\pi^*,u^*}(s), \lambda(s))\lambda(s)ds\Big]. \end{aligned} \tag{7.2}$$

The last term in (7.2) arises since there might be a jump in the value function $v^*(T\wedge\tau_n, X^{\pi^*,u^*}(T\wedge\tau_n), \lambda(T\wedge\tau_n))\mathbf{1}\{\tau > T\wedge\tau_n\}$ and the jump in the value function arrives with the intensity $\lambda(t)$ (which is the mortality intensity for $\tau$). Recalling the HJB equation (3.14), using the property of conditional expectations and changing the order of integration, we derive

$$
\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*(T\wedge\tau_n, X^{\pi^*,u^*}(T\wedge\tau_n),\lambda(T\wedge\tau_n))\mathbf{1}\{\tau>T\wedge\tau_n\}\Big] = v^*(t,x,\lambda)\\
&\quad+\mathbb{E}_{t,x,\lambda}\Big[-\int_t^{T\wedge\tau_n\wedge\tau}\Big(\int_s^{T\wedge\tau}\phi'(w-s)(u^*(w))^\gamma dw + q\phi'(T-s)(X^{\pi^*,u^*}(T))^\gamma\mathbf{1}_{\{\tau>T\}}\Big)ds\\
&\quad-\int_t^{T\wedge\tau_n\wedge\tau}(u^*(s))^\gamma ds\Big] = v^*(t,x,\lambda)+\mathbb{E}_{t,x,\lambda}\Big[-\int_t^{T\wedge\tau_n\wedge\tau}\phi(w-t)(u^*(w))^\gamma dw\\
&\quad+\int_{T\wedge\tau_n\wedge\tau}^{T\wedge\tau}\big(\phi(w-(T\wedge\tau_n\wedge\tau))-\phi(w-t)\big)(u^*(w))^\gamma dw\\
&\quad+q\big(\phi(T-(T\wedge\tau_n))-\phi(T-t)\big)(X^{\pi^*,u^*}(T))^\gamma\mathbf{1}\{\tau>T\}\Big].
\end{aligned}
$$

We now take $n\to\infty$. Applying Lebesgue's dominated convergence theorem (recall that we assume that the family $\big\{v^*(\mathcal{T},X^{\pi^*,u^*}(\mathcal{T}),\lambda(\mathcal{T})),\mathcal{T}\ \textit{is an}\ \mathbb{F}-\textit{stopping time}\big\}$ is uniformly integrable) and using the terminal condition for $v^*(T,x,\lambda)$ from the HJB equation (3.14), we can prove (7.1).

2. We now prove that the strategy $(\pi^*,u^*)$ defined by (3.14)-(3.15) is an equilibrium strategy. Fix $(t,x)\in[0,T)\times\mathbb{R}$ and choose $(\pi_h,u_h)\in\mathcal{A}$ as specified in Definition 3.3. We have

$$
\begin{aligned}
&v^{\pi_h,u_h}(t,x,\lambda)\\
&\quad= v^{\pi_h,u_h}(t,x,\lambda)-\mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]\\
&\qquad+\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big],
\end{aligned}\tag{7.3}
$$

since $(\pi^*,u^*)$ is applied after time $s\geq t+h$ and $v^*$ is the value function corresponding to $(\pi^*,u^*)$, as proved in the previous point. Equation (7.3) can be rewritten as

$$
\begin{aligned}
&v^{\pi_h,u_h}(t,x,\lambda)-v^*(t,x,\lambda)\\
&\quad=\Big(v^{\pi_h,u_h}(t,x,\lambda)-\mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]\Big)\\
&\qquad+\Big(\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]-v^*(t,x,\lambda)\Big).
\end{aligned}\tag{7.4}
$$

We deal with the first term in (7.4). Recalling the definition of the value function $v^{\pi,u}$, see (3.10), we immediately get

$$\begin{aligned}
&v^{\pi_h,u_h}(t,x,\lambda)\\
&= \mathbb{E}_{t,x,\lambda}\left[\int_t^{\tau\wedge T}\phi(s-t)(u^*(s))^\gamma ds + q\phi(T-t)(X^{\pi_h,u_h}(T))^\gamma \mathbf{1}_{\{\tau>T\}}\right],
\end{aligned}$$

and using the property of conditional expectations we derive

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]\\
&= \mathbb{E}_{t,x,\lambda}\left[\int_{\tau\wedge(t+h)}^{\tau\wedge T}\phi(s-t-h)(u^*(s))^\gamma ds + q\phi(T-t-h)(X^{\pi_h,u_h}(T))^\gamma \mathbf{1}_{\{\tau>T\}}\right].
\end{aligned}$$

Hence, the first term in (7.4) is equal to

$$\begin{aligned}
&v^{\pi_h,u_h}(t,x,\lambda) - \mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]\\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge(t+h)}\phi(s-t)(u(s))^\gamma ds\Big] + \mathbb{E}_{t,x,\lambda}\Big[\int_{\tau\wedge(t+h)}^{\tau\wedge T}(\phi(s-t)-\phi(s-t-h))(u^*(s))^\gamma ds\Big]\\
&\quad + \mathbb{E}_{t,x,\lambda}\Big[q(\phi(T-t)-\phi(T-t-h))(X^{\pi_h,u_h}(T))^\gamma 1_{\{\tau>T\}}\Big] \qquad (7.5)
\end{aligned}$$

We now investigate three terms on the right hand side in (7.5). If we apply Fubini's theorem and differentiate the Lebesgue's integral, we obtain

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[\int_t^{t+h}\phi(s-t)(u(s))^\gamma \mathbf{1}\{s\le\tau\}ds\Big]}{h}\\
&= \lim_{h\to 0}\frac{\int_t^{t+h}\mathbb{E}_{t,x,\lambda}\big[\phi(s-t)(u(s))^\gamma \mathbf{1}\{s\le\tau\}\big]ds}{h} = (u(t))^\gamma. \qquad (7.6)
\end{aligned}$$

Since

$$\begin{aligned}
|\phi(s-t)-\phi(s-t-h)| &\le Kh,\\
\frac{\int_t^{t+h}|\phi(s-t)-\phi(s-t-h)|(u^*(s))^\gamma ds}{h} &\le K\int_t^{t+h}(u^*(s))^\gamma ds,
\end{aligned}$$

we can apply Lebesgue's dominated convergence theorem and we can derive the limit

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[\int_{\tau\wedge(t+h)}^{\tau\wedge T}(\phi(s-t)-\phi(s-t-h))(u^*(s))^\gamma ds\Big]}{h}\\
&= \lim_{h\to 0}\mathbb{E}_{t,x,\lambda}\Big[\int_t^T\frac{\phi(s-t)-\phi(s-t-h)}{h}(u^*(s))^\gamma\mathbf{1}\{s\le\tau\}ds\mathbf{1}\{\tau>t+h\}\Big]\\
&\quad -\lim_{h\to 0}\mathbb{E}_{t,x,\lambda}\Big[\frac{\int_t^{t+h}(\phi(s-t)-\phi(s-t-h))(u^*(s))^\gamma ds}{h}\mathbf{1}\{\tau>t+h\}\Big]\\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\phi'(s-t)(u^*(s))^\gamma ds\Big]. \qquad (7.7)
\end{aligned}$$

Before we prove the limit for the third term in (7.5) we need some preliminary results. Let $X^{t,x,\pi_h,u_h}$ denote the wealth process which starts at $x$ at time $t$. We can notice that $X^{t,x,\pi_h,u_h}(T) = X^{t+h,X^{t,x,\pi_h,u_h}(t+h),\pi^*,u^*}(T)$. By continuity of integrals in (3.5) we can conclude that $X^{t,x,\pi_h,u_h}(t+h)\to x, h\to 0$. By classical results on SDEs, see e.g. Theorem II.5.2 in Kunita (1984) or Becherer and Schweizer (2005), we know that $(t,x)\mapsto X^{t,x,\pi_h,u_h}$ is continuous. Hence, we can conclude that $X^{t+h,X^{t,x,\pi_h,u_h}(t+h),\pi^*,u^*}(T)\to X^{t,x,\pi^*,u^*}(T),\ h\to 0$. By inequality (3.6) and admissability of $(\pi_h,u_h)$ we obtain the estimate

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\frac{\phi(T-t)-\phi(T-t-h)}{h}(X^{t,x,\pi_h,u_h}(T))^\gamma\Big|^2\Big]\le K\Big(1+\mathbb{E}\Big[\int_t^{t+h}|\pi(s)X^{\pi,u}(s)|^2ds\\
&\quad+\int_t^{t+h}|u(s)|^2ds+\int_{t+h}^T|\pi^*(s)X^{t+h,X^{t,x,\pi,u}(t+h),\pi^*,u^*}(s)|^2ds+\int_{t+h}^T|u^*(s)|^2ds\Big]\Big)\le K<\infty,
\end{aligned}$$

and we can deduce that the sequence $\big\{\frac{\phi(T-t)-\phi(T-t-h)}{h}(X^{\pi_h,u_h}(T))^\gamma, h\in[0,\epsilon]\big\}$ is uniformly integrable. Now we can establish the limit

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[q(\phi(T-t)-\phi(T-t-h))(X^{\pi_h,u_h}(T))^\gamma\mathbf{1}_{\{\tau>T\}}\Big]}{h}\\
&= \lim_{h\to 0}\mathbb{E}_{t,x,\lambda}\Big[q\frac{\phi(T-t)-\phi(T-t-h)}{h}(X^{\pi_h,u_h}(T))^\gamma\mathbf{1}_{\{\tau>T\}}\Big]\\
&= \mathbb{E}_{t,x,\lambda}\Big[q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma 1_{\{\tau>T\}}\Big]. \qquad (7.8)
\end{aligned}$$

Collecting (7.6)-(7.8), we get

$$\lim_{h\to 0}\frac{v^{\pi_h,u_h}(t,x,\lambda)-\mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]}{h}$$
$$= (u(t))^\gamma+\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\phi'(s-t)(u^*(s))^\gamma ds+q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma\mathbf{1}_{\{\tau>T\}}\Big] \quad (7.9)$$

We deal with the second term in (7.4). Recall that an arbitrary $(\pi,u)$ is applied on $[t,t+h]$. We introduce a localizing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n\to T,\ n\to\infty$. Applying Itô's formula, as in point 1, we can derive

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*((t+h)\wedge\tau_n,X^{\pi_h,u_h}((t+h)\wedge\tau_n),\lambda((t+h)\wedge\tau_n))\mathbf{1}\{\tau>(t+h)\wedge\tau_n\}\Big]-v^*(t,x,\lambda)\\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{(t+h)\wedge\tau_n\wedge\tau}v_t^*(s,X^{\pi,u}(s),\lambda(s))ds+\int_t^{(t+h)\wedge\tau_n\wedge\tau}\mathcal{L}_x^{\pi,u}v^*(s,X^{\pi,u}(s),\lambda(s))ds\\
&\quad+\int_t^{(t+h)\wedge\tau_n\wedge\tau}\mathcal{L}_\lambda v^*(s,X^{\pi,u}(s),\lambda(s))ds-\int_t^{(t+h)\wedge\tau_n\wedge\tau}v^*(s,X^{\pi,u}(s),\lambda(s))\lambda(s)ds\Big]\\
&\le \mathbb{E}_{t,x,\lambda}\Big[\int_t^{(t+h)\wedge\tau_n\wedge\tau}\sup_{\pi,u}\Big\{(u(s))^\gamma+v_t^*(s,X^{\pi,u}(s),\lambda(s))+\mathcal{L}_x^{\pi,u}v^*(s,X^{\pi,u}(s),\lambda(s))\\
&\quad+\int_t^{(t+h)\wedge\tau_n\wedge\tau}\mathcal{L}_\lambda v^*(s,X^{\pi,u}(s),\lambda(s))ds-v^*(s,X^{\pi,u}(s),\lambda(s))\lambda(s)\Big\}ds\\
&\quad-\int_t^{(t+h)\wedge\tau_n\wedge\tau}(u(s))^\gamma ds\Big],
\end{aligned}$$

where the supremum is with respect to $u^\gamma$ and $\mathcal{L}_x^{\pi,u}$. Since the function $v^*$ satisfies the HJB equation (3.14) we get the inequality

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*((t+h)\wedge\tau_n,X^{\pi_h,u_h}((t+h)\wedge\tau_n),\lambda((t+h)\wedge\tau_n))\mathbf{1}\{\tau>(t+h)\wedge\tau_n\}\Big]-v^*(t,x,\lambda)\\
&\le \mathbb{E}_{t,x,\lambda}\Big[-\int_t^{(t+h)\wedge\tau_n\wedge\tau}\Big(\int_s^{\tau\wedge T}\phi'(w-t)(u^*(w))^\gamma dw+q\phi'(T-s)(X^{\pi^*,u^*}(T))^\gamma 1_{\{\tau>T\}}\Big)ds\\
&\quad-\int_t^{(t+h)\wedge\tau_n\wedge\tau}(u(s))^\gamma ds\Big] \qquad (7.10)
\end{aligned}$$

We now take $n\to\infty$. We apply Fatou's lemma on the left hand side of (7.10) together with Lebesgue's dominated convergence theorem on the right hand side (7.10) and we derive the inequality

$$
\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h))\mathbf{1}\{\tau > t+h\}\Big] - v^*(t,x,\lambda) \\
&\quad \le \ \mathbb{E}_{t,x,\lambda}\Big[-\int_t^{(t+h)\wedge\tau}\Big(\int_s^{\tau\wedge T}\phi'(w-t)(u^*(w))^\gamma dw + q\phi'(T-s)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}_{\{\tau>T\}}\Big)ds \\
&\qquad -\int_t^{(t+h)\wedge\tau}(u(s))^\gamma ds\Big].
\end{aligned}
$$

If we use Fubini's theorem and differentiate the Lebesque's integral, we obtain the inequality

$$
\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h))\mathbf{1}\{\tau > t+h\}\Big] - v^*(t,x,\lambda)}{h} \\
&\quad \le \ -\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\phi'(w-t)(u^*(w))^\gamma dw + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma 1_{\{\tau>T\}}\Big] \\
&\qquad -(u(t))^\gamma.
\end{aligned} \tag{7.11}
$$

Finally, we substitute (7.9) and (7.11) into (7.4) and we prove that $(\pi^*, u^*)$ is an equilibrium strategy, i.e. $(\pi^*, u^*)$ satisfies the inequality (3.13). □

**Proof of Proposition 3.1:**

The proof is divided into several steps.

1. *We prove that there exists a unique solution* $f \in \mathcal{C}([0,T]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,T)\times\mathcal{K})$ *to the PDE* (4.11) *and that* $0 < K_l \le f(t,\lambda) \le K_u$. We define the operator

$$
\begin{aligned}
(\mathcal{M}f)(t,\lambda) = \mathbb{E}_{t,\lambda}\Big[&q e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(T-t) - \int_t^T \lambda(u)du} \\
&+ \int_t^T (1-\gamma)e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t) - \int_t^s \lambda(u)du}\big(f(s,\lambda(s))\big)^{\frac{\gamma}{\gamma-1}}ds\Big].
\end{aligned} \tag{7.12}
$$

Recall that the process $\lambda$ is assumed to be bounded from below. It is easy to see that if $f(t,\lambda) \ge K_l > 0$ then $f^*(t,\lambda) = (\mathcal{M}f)(t,\lambda) \ge K_l$ (the constant $K_l$ is determined by the first term in (7.12)). Moreover, if $f(t,\lambda) \ge K_l > 0$ and $f(t,\lambda) \le K_u < +\infty$

then $f^*(t,\lambda) = (\mathcal{M}f)(t,\lambda) \le K_u$ (the constant $K_u$ is determined by both terms in (7.12) and the lower bound $K_l$). Hence, the non-linear term in the PDE (4.11), i.e. the mapping $f \mapsto f^{\frac{\gamma}{\gamma-1}}$ is Lipschitz continuous on $[K_l, K_u]$. By Propositions 2.1 and 2.3 from Becherer and Schweizer (2005) the operator $\mathcal{M}$ is a contraction in the class of functions $\{f : 0 < K_l \le f(t,\lambda) \le K_u < +\infty\}$ and the fixed point of the operator $f^*(t,\lambda) = (\mathcal{M}f^*)(t,\lambda)$ is the unique solution to the PDE (4.11). Moreover, $f^* \in \mathcal{C}([0,T]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,T)\times\mathcal{K})$.

2. *We prove that $f$ is Lipschitz continuous in $\lambda$ uniformly in $t$.* Let us consider a sequence $f_{n+1}(t,\lambda) = \mathcal{M}f_n(t,\lambda)$. We know that $f_n \to f,\ n\to\infty$, by the previous point. Let us initiate the iteration with a function $f_0$ which satisfies the Lipschitz condition

$$|f_0(t,\lambda) - f_0(t,\lambda')| \le \theta(t)|\lambda-\lambda'|,$$

where $\theta$ will be specified in the sequel. Assume that

$$|f_n(t,\lambda) - f_n(t,\lambda')| \le \theta(t)|\lambda-\lambda'|, \tag{7.13}$$

for some $n\in\mathbb{N}$. Clearly, inequality (7.13) holds for $n=0$. We now show that inequality (7.13) holds for $n+1$. Let $(\lambda^{t,\lambda}(s), t \le t \le T)$ denote the process which solves the SDE (2.4) and starts at time $t$ from $\lambda(t)=\lambda$. Recalling the definition of the operator from (7.12) we can derive

$$\begin{aligned}
&|f_{n+1}(t,\lambda) - f_{n+1}(t,\lambda')|\\
&\quad\le\ K\Big(\mathbb{E}\Big[\int_t^T |\lambda^{t,\lambda}(s) - \lambda^{t,\lambda'}(s)|ds + \int_t^T |f_n(s,\lambda^{t,\lambda}(s)) - f_n(s,\lambda^{t,\lambda'}(s))|ds\Big]\Big)\\
&\quad\le\ K\Big(1+\int_t^T \theta(s)ds\Big)\mathbb{E}\Big[\sup_{s\in[t,T]}|\lambda^{t,\lambda}(s) - \lambda^{t,\lambda'}(s)|\Big]\\
&\quad\le\ K\Big(1+\int_t^T \theta(s)ds\Big)|\lambda-\lambda'|,
\end{aligned}\tag{7.14}$$

where we use some standard estimates based on the mean-value theorem, Lipschitz continuity of the mapping $f \mapsto f^{\frac{\gamma}{\gamma-1}}$ on $[K_l, K_u]$, uniform boundedness of the process $\lambda$, uniform boundedness of the sequence $f_n$, inequality (7.13) and a classical estimate for $\mathbb{E}\big[\sup_{s\in[t,T]}|\lambda^{t,\lambda}(s)-\lambda^{t,\lambda'}(s)|\big]$ from the theory of SDEs, see e.g equation (4.6) in El Karoui et al. (1997). Let us choose $\theta$ such that

$$\theta(t) = K\Big(1 + \int_t^T \theta(s)ds\Big),$$

i.e. we choose $\theta(t) = Ke^{K(T-t)}$ with a sufficiently large $K$. With such a choice of $\theta$, from (7.14) we immediately get the inequality

$$|f_{n+1}(t,\lambda) - f_{n+1}(t,\lambda')| \le \theta(t)|\lambda - \lambda'|. \tag{7.15}$$

Since the function $\theta$ in (7.15) does not depend on $n$, we conclude that the limit $f_n \to f,\ n \to \infty$, is also Lipschitz continuous in $\lambda$ uniformly in $t$.

3. *We prove that $f$ is Hölder continuous in $t$ uniformly in $\lambda$.* Let us consider the BSDE

$$\begin{aligned} dY^{t,\lambda}(s) &= \Big(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2} + \lambda^{t,\lambda}(s)\Big)Y^{t,\lambda}(s)ds \\ &\quad -(1-\gamma)f(s,\lambda^{t,\lambda}(s))^{\frac{\gamma}{\gamma-1}}ds + Z^{t,\lambda}(s)dW_\lambda(s), \quad 0 \le s \le T, \\ Y^{t,\lambda}(T) &= q, \end{aligned} \tag{7.16}$$

and the forward dynamics of the process $(\lambda^{t,\lambda}(s), t \le s \le T)$ is given by the SDE (2.4). Recall that $\lambda^{t,\lambda}$ denotes the process $\lambda$ which starts at time $t$ from $\lambda(t) = \lambda$. Since $\lambda$ and $f$ are bounded, we deal with a linear BSDE. By Theorem 2.1 in El Karoui et al. (1997) there exists a unique solution to the BSDE (7.16) and the solution has the probabilistic representation $Y^{t,\lambda}(s) = (\mathcal{M}f)(s,\lambda^{t,\lambda}(s))$. Moreover, by uniqueness of solution to the BSDE (7.16) we must have $Y^{t,\lambda}(s) = f(s,\lambda^{t,\lambda}(s))$. We notice that the generator of the BSDE (7.16) is bounded and Lipschitz continuous in $\lambda$ uniformly in $(t,y)$ since the process $Y$ is bounded and $f$ together with $f^{\frac{\gamma}{\gamma-1}}$ are Lipschitz continuous in $\lambda$ (as proved in the previous point). Hence, by Proposition 4.1 from El Karoui et al. (1997) we have the inequality

$$|f(t,\lambda)-f(t',\lambda)|\le K|t-t'|^{1/2},$$

and our assertion is proved.

4. *We prove that for any* $s\in[0,T]$ *there exists a unique solution* $P^s\in\mathcal{C}([0,s]\times\mathcal{K})\cap\mathcal{C}^{1,2}([0,s)\times\mathcal{K})$ *to the PDE* (4.15). *The solution* $P^s$ *is Lipschitz continuous in* $\lambda$ *uniformly in* $t$ *and Hölder continuous in* $t$ *uniformly in* $\lambda$. Since $\lambda$ and $f$ are bounded and $f(t,\lambda), f^{\frac{\gamma}{\gamma-1}}(t,\lambda), f^{\frac{1}{\gamma-1}}(t,\lambda)$ are uniformly Hölder continuous in $(t,\lambda)$, by Proposition 2.3 from Becherer and Schweizer (2005) there exists a unique solution $P^s\in\mathcal{C}([0,s]\times\mathcal{K})\cap\mathcal{C}^{1,2}([0,s)\times\mathcal{K})$ to the PDE (4.15) for any $s\in[0,T]$. Moreover, we have the representation

$$\begin{aligned}&P^s(t,\lambda)\\ &=\ \mathbb{E}_{t,\lambda}\Big[e^{-\left(\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t)-\int_t^s\left(\lambda(u)+\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}\right)du}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\Big],\quad 0\le t\le s.\end{aligned}\tag{7.17}$$

Let us now consider the BSDE

$$\begin{aligned}dY^{t,\lambda}(u)&=\Big(\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}+\lambda^{t,\lambda}(u)\\&\quad+\gamma f(t,\lambda^{t,\lambda}(u))^{\frac{1}{\gamma-1}}\Big)Y^{t,\lambda}(u)du+Z^{t,\lambda}(u)dW_\lambda(u),\quad 0\le u\le s,\\ Y^{t,\lambda}(s)&=f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}},\end{aligned}\tag{7.18}$$

and the forward dynamics of the process $(\lambda^{t,\lambda}(s),t\le s\le T)$ given by the SDE (2.4). As in the previous point, we can prove our assertion.

5. *We prove that there exists a unique solution* $F\in\mathcal{C}([0,T]\times\mathcal{K})\cap\mathcal{C}^{1,2}([0,T)\times\mathcal{K})$ *to the PDE* (4.16) *and that* $K_l\le F(t,\lambda)\le 0$. Using our previous results and representation (4.14) for the function $Q$ we can prove that the function $Q$ is uniformly

Hölder continuous in $(t,\lambda)$. The assertion of this point and the representation

$$F(t,\lambda) = \mathbb{E}_{t,x,\lambda}\Big[\int_t^T Q(s,\lambda(s))e^{-\left(\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t)-\int_t^s\left(\lambda(u)+\gamma f(u,\lambda(u))^{1/(\gamma-1)}\right)du}ds\Big], \quad (7.19)$$

follow now from Proposition 2.3 in Becherer and Schweizer (2005). From (4.14) we conclude that the sign of $Q$, and consequently the sign of $F$, is determined by the sign of the derivative $\vartheta'$. Since

$$\vartheta'(t) = -\frac{\delta}{1+\delta t} \leq 0, \quad 0 \leq t \leq T,$$

the function $F$ is non-positive. $\square$

**The proof of representation** (4.13)**-**(4.14)**:** Since $v^0(t,x,\lambda) = (x+g(t))^\gamma f(t,\lambda)$, from (4.10) we get the formulas

$$\pi^{0,*}(t,x,\lambda) = \frac{\mu-r}{\sigma^2(1-\gamma)}\frac{x+g(t)}{x}, \quad u^{0,*}(t,x,\lambda) = (x+g(t))f(t,\lambda)^{\frac{1}{\gamma-1}}.$$

Recall that $g(T)=0$ and $f(T,\lambda)=q$. It is easy to see that in order to calculate the expectation in (4.13) we have to calculate the expectation

$$\mathbb{E}_{t,x,\lambda}\big[e^{-\rho(s-t)}(X^{\pi^{0,*},u^{0,*}}(s)+g(s))^\gamma f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\mathbf{1}\{s<\tau\}\big], \quad t \leq s \leq T. \quad (7.20)$$

Using similar techniques that lead to the SDE (4.21), we can deduce the dynamics

$$\begin{aligned} d(X^{\pi^{0,*},u^{0,*}}(s)+g(s)) &= (X^{\pi^{0,*},u^{0,*}}(t)+g(s))dZ(s), \quad t \leq s \leq T, \\ dZ(s) &= \big(\tilde{\mu} - f(s,\lambda(s))^{\frac{1}{\gamma-1}})\big)ds + \tilde{\sigma}dW_m(s), \quad t \leq s \leq T, \end{aligned}$$

and, as in (4.22), we get the solution

$$\begin{aligned}&X^{\pi^{0,*},u^{0,*}}(s)+g(s)\\ = & \ (X^{\pi^{0,*},u^{0,*}}(t)+g(t))e^{(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)(s-t)-\int_t^s f(u,\lambda(u))^{\frac{1}{\gamma-1}}du+\tilde{\sigma}(W_m(s)-W_m(t))}, \quad t\leq s\leq T.\end{aligned}$$

By independence of $(\lambda,\tau)$ and $W$ and the property of conditional expectation we derive

$$\begin{aligned}&\mathbb{E}_{t,x,\lambda}\Big[e^{-\rho(s-t)}\big(X^{\pi^{0,*},u^{0,*}}(s)+g(s)\big)^{\gamma}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\mathbf{1}\{s<\tau\}\Big]\\ = & \ \big(x+g(t)\big)^{\gamma}\mathbb{E}_{t,x,\lambda}\Big[e^{-\rho(s-t)+\gamma(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)(s-t)+\gamma\tilde{\sigma}(W_m(s)-W_m(t))}\Big]\\ & \cdot\mathbb{E}_{t,x,\lambda}\Big[e^{-\int_t^s\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}du}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\mathbb{E}_{t,x,\lambda}\big[\mathbf{1}\{s<\tau\}|\sigma(\lambda(s),s\in[t,T])\big]\Big]\\ = & \ \big(x+g(t)\big)^{\gamma}e^{-\rho(s-t)+\gamma(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)(s-t)+\frac{1}{2}\gamma^2\tilde{\sigma}^2(s-t)}\\ & \cdot\mathbb{E}_{t,\lambda}\Big[e^{-\int_t^s\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}du-\int_t^s\lambda^{t,\lambda}(u)du}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\Big], \quad t\leq s\leq T. \qquad (7.21)\end{aligned}$$

Since $g(T)=0$ and $f(T,\lambda)=q$, we get representation (4.14) with function $P^s$ defined in (7.17). The fact that $P^s$ satisfies the PDE (4.15) is proved while proving Proposition 3.1. □

**The proof of formula** (4.9)**:** We follow the idea from Dong and Sircar (2014). We assume that the following expansions hold:

$$\begin{aligned}v(t,x,\lambda) &= v^0(t,x,\lambda)+v^1(t,x,\lambda)\varepsilon+o(\varepsilon^2),\\ u^*(t) &= u^{0,*}(t)+u^{1,*}(t)\varepsilon+o(\varepsilon^2),\\ \pi^*(t) &= \pi^{0,*}(t)+\pi^{1,*}(t)\varepsilon+o(\varepsilon^2),\\ X^{\pi^*,u^*}(t) &= X^{\pi^{0,*},u^{0,*}}(t)+X^{\pi^{1,*},u^{1,*}}(t)\varepsilon+o(\varepsilon^2),\end{aligned}$$

where $v$ denotes the equilibrium value function under an equilibrium strategy $(\pi^*,u^*)$. The first order approximations are to be determined. Let us recall that we consider Markov strategies, i.e. $u(t)=u(t,X^{\pi,u}(t),\lambda(t))$. In the sequel we assume $u^*,u^{0,*},u^{1,*}$

are differentiable with respect to the state variable $x$. We have

$$
\begin{aligned}
(u^*(t))^\gamma = (u^*(X^{\pi^*,u^*}(t)))^\gamma = \Big(u^*\big(X^{\pi^{0,*},u^{0,*}}(t) + X^{\pi^{1,*},u^{1,*}}(t)\varepsilon + o(\varepsilon^2)\big)\Big)^\gamma \\
= \Big(u^{0,*}\big(X^{\pi^{0,*},u^{0,*}}(t) + X^{\pi^{1,*},u^{1,*}}(t)\varepsilon\big) + u^{1,*}\big(X^{\pi^{0,*},u^{0,*}}(t) + X^{\pi^{1,*},u^{1,*}}(t)\varepsilon\big)\varepsilon + o(\varepsilon^2)\Big)^\gamma \\
= \Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t)) + \Big((u^{0,*})'(X^{\pi^{0,*},u^{0,*}}(t))X^{\pi^{1,*},u^{1,*}}(t) + u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t))\Big)\varepsilon + o(\varepsilon^2)\Big)^\gamma \\
= \big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^\gamma \\
+\gamma\big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^{\gamma-1}\Big((u^{0,*})'(X^{\pi^{0,*},u^{0,*}}(t))X^{\pi^{1,*},u^{1,*}}(t) + u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t))\Big)\varepsilon + o(\varepsilon^2) \\
= \big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^\gamma + R_1\Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t)), u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t)), X^{\pi^{1,*},u^{1,*}}(t)\Big)\varepsilon + o(\varepsilon^2),
\end{aligned}
$$

with properly defined $R_1$. Similarly, we get

$$
\begin{aligned}
(X^{\pi^*,u^*}(T))^\gamma = \Big(X^{\pi^{0,*},u^{0,*}}(T) + X^{\pi^{1,*},u^{1,*}}(T)\varepsilon + o(\varepsilon^2)\Big)^\gamma \\
= (X^{\pi^{0,*},u^{0,*}}(T))^\gamma + \gamma(X^{\pi^{0,*},u^{0,*}}(T))^{\gamma-1}X^{\pi^{1,*},u^{1,*}}(T)\varepsilon + o(\varepsilon^2) \\
= (X^{\pi^{0,*},u^{0,*}}(T))^\gamma + R_2\Big(X^{\pi^{0,*},u^{0,*}}(T), X^{\pi^{1,*},u^{1,*}}(T)\Big)\varepsilon + o(\varepsilon^2),
\end{aligned}
$$

with properly defined $R_2$. Recall now the definition of the value function (3.10) and the discounting function (4.2). Using the expansions for $u^*$ and $X^{\pi^*,u^*}$ we derive

$$
\begin{aligned}
v(t,x,\lambda) = v^{\pi^*,u^*}(t,x,\lambda) = \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} e^{-\rho(s-t)}(1+\vartheta(s-t)\varepsilon) \\
\cdot\Big(\big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(s))\big)^\gamma + R_1\Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(s)), u^{1,*}(X^{\pi^{0,*},u^{0,*}}(s)), X^{\pi^{1,*},u^{1,*}}(s)\Big)\varepsilon\Big)ds \\
+qe^{-\rho(T-t)}(1+\vartheta(T-t)\varepsilon) \\
\cdot((X^{\pi^{0,*},u^{0,*}}(T))^\gamma + R_2\Big(X^{\pi^{0,*},u^{0,*}}(T), X^{\pi^{1,*},u^{1,*}}(T)\Big)\varepsilon)\mathbf{1}\{\tau > T\}\Big] + o(\varepsilon^2). \qquad (7.22)
\end{aligned}
$$

We can now identify functions $v^0$ and $v^1$ so that

$$
v(t,x,\lambda) \;=\; v^0(t,x,\lambda) + v^1(t,x,\lambda)\varepsilon + o(\varepsilon^2). \qquad (7.23)
$$

In particular, we see that $v^0$ is the value function under the strategy $(\pi^{0,*}, u^{0,*})$ for the optimization problem (3.10) with exponential discounting $\rho$.

We are now ready to handle the non-local term in the HJB (4.4). We use the expansion

$$\phi'(u) = \phi(u)(-\rho + \varepsilon\vartheta'(u)) = e^{-\rho u}(-\rho + (\vartheta'(u) - \rho\vartheta(u))\varepsilon) + o(\varepsilon^2), \quad \varepsilon \to 0.$$

The non-local term in the HJB takes the form

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\}\Big] \\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} e^{-\rho(s-t)}(-\rho + (\vartheta'(s-t) - \rho\vartheta(s-t))\varepsilon) \\
&\quad \cdot\Big(\big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^\gamma \\
&\quad\quad + R_1\Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t)), u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t)), X^{\pi^{1,*},u^{1,*}}(t)\Big)\varepsilon\Big) ds \\
&\quad + qe^{-\rho(T-t)}(-\rho + (\vartheta'(T-t) - \rho\vartheta(T-t))\varepsilon) \\
&\quad \cdot\Big((X^{\pi^{0,*},u^{0,*}}(T))^\gamma + R_2\Big(X^{\pi^{0,*},u^{0,*}}(T), X^{\pi^{1,*},u^{1,*}}(T)\Big)\varepsilon\Big)\mathbf{1}\{\tau > T\}\Big] + o(\varepsilon^2)
\end{aligned}$$

Collecting the terms independent of $\varepsilon$, the terms proportional to $\varepsilon$ and the terms of order $o(\varepsilon^2)$, recalling $v^0$ and $v^1$ identified in (7.22), we get the key expansion

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\}\Big] \\
&= -\rho v^0(t,x,\lambda) - \rho v^1(t,x,\lambda)\varepsilon \\
&\quad + \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds \\
&\quad + q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma \mathbf{1}\{\tau > T\}\Big]\varepsilon + o(\varepsilon^2), \qquad (7.24)
\end{aligned}$$

which completes the proof. □

**Proof of Theorem 4.2:**

Recall that $X^{\tilde{\pi}^*,\tilde{u}^*}(t) + g(t)$ is a stochastic exponential of the form (4.22). By boundedness of $\tilde{f}$ and Doob's inequality we get the estimate

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)|^2\Big]<\infty. \tag{7.25}$$

The uniform integrability follows from estimate (7.25) and boundedness of $f$ and $F$. $\square$

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